Chapter 2: Combinatorial Probability

Section 2.1: Permutations and Combinations

Example: The Chipotle problem

Menu item choice: burrito, bowl, salad, crispy corn tacos, soft corn tacos, soft flour tacos

Rice choice: white, brown

Bean choice: black, pinto

Grilled veggies: yes, no

Protein choice: steak, chicken, carnitas, barbacoa, sofritas, chorizo

Condiments: mild, medium, + hot salsa, sour cream, pico de gallo, corn salsa, lettuce, guac, + cheese

How many different meals can you make?

$$\underset{\text{menu items}}{6} \cdot \underset{\text{rice choices}}{2} \cdot \underset{\text{bean choices}}{2} \cdot \underset{\text{veg choices}}{2} \cdot \underset{\text{protein choices}}{6} \cdot \underset{\text{cond. choices}}{9} = 36 \cdot 8 \cdot 9 = 2592 \text{ ways!}$$

This isn't counting menu tricks... or ordering more than one condiment...

The Multiplication Rule

Suppose m tasks are carried out so that, regardless of the outcomes of the other experiments, experiment i has n_i possible outcomes. Then the total # of possible outcomes is

$$n_1 n_2 n_3 \cdots n_i \cdots n_m$$

EX:

How many ways can 5 people stand in a line?

$$\underset{\text{pos. 1}}{5} \cdot \underset{\text{pos. 2}}{4} \cdot 3 \cdot 2 \cdot 1 = 5!$$

Ex: Suppose you've entered a raffle, along with 199 other people. There are 3 raffle prizes. How many combinations of people could be drawn in the raffle?

$$100 \cdot 99 \cdot 98 = 970{,}200$$

In general, if we have k raffle items and n people enter, the answer is

$$n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-k+1)$$

We say there are this many 'permutations' of people

~~This product is called the number of permutations of k objects from a set of size n.~~

Def:

The number of permutations of k objects from a set of size n is given by

$$P_{n,k} = n(n-1)(n-2)\cdots(n-k+1)\left(\frac{(n-k)!}{(n-k)!}\right) = \frac{n!}{(n-k)!}$$

Example: (worksheet)

Fred is going to dinner M-F, with each dinner being at one of his favorite restaurants. But he doesn't want to go to the same place twice! How many ways can we do this?

$$P_{10,5} = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = \frac{10!}{(10-5)!} = \frac{10!}{5!} = 30{,}240$$

Note:

Frequently, we call something a permutation when the order matters.

Example:

How many subsets of of $\Omega = \{a,b,c\}$ are there?

$\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a,b\}$, $\{b,c\}$, $\{a,c\}$, $\{a,b,c\}$ $\Rightarrow 8$

Definition:

The number of distinct subsets w/ k elements that can be drawn from a set with n elements is given by

$$C_{n,k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \begin{array}{l} \leftarrow \text{possible choices} \\ \leftarrow \text{possible orderings} \end{array}$$

$$= \frac{n!}{k!(n-k)!}$$

i.e., the # of combinations of k elements of a set of size n.

Note:

$C_{n,k}$ is aka the binomial coefficient, and is frequently written as

$$C_{n,k} = \underbrace{\binom{n}{k}}_{\text{"n choose k"}}$$

Check:

For subsets of $\Omega = \{a,b,c\}$, we have

$$\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 1 + 3 + 3 + 1 = 8$$

Example:

Suppose we flip 5 coins. What is the # of outcomes with 0, 1, or 2 heads?

$|\Omega| = 2^5 = 32$

0 heads: T T T T T $\rightarrow$ $1 = \binom{5}{0} = \frac{5!}{5!0!}$

1 head: 5 outcomes $\rightarrow$ $5 = \binom{5}{1} = \frac{5!}{4!1!}$

2 heads: $\binom{5}{2} = \frac{5!}{3!2!} = \frac{5 \cdot 4}{2} = 10$

↑ # ways to pick 2 tosses for the head to occur

Notice that this is symmetric:

3 heads: $\binom{5}{3} = \frac{5!}{3!\,2!} = 10$

4 heads: $\binom{5}{4} = 5$

5 heads $\binom{5}{5} = 1$

In general,

$$\binom{n}{m} = \binom{n}{n-m}$$

If we visualize what happens when we do this experiment for 1, 2, 3, ... coins, we get Pascal's Triangle

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

Notice that each number is the sum of the nearest two on the row immediately above it:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Proof: (on worksheet)

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(n-k)!\,(k-1)!} + \frac{(n-1)!}{(n-k-1)!\,k!}$$

$$= \frac{(n-1)!\,k + (n-1)!\,(n-k)}{(n-k)!\,k!}$$

$$= \frac{n!}{(n-k)!\,k!}$$

$$= \binom{n}{k}$$

Furthermore, the numbers in Pascal's Δ are the coefficients in the Binomial Theorem:

$$(x+y)^n = \sum_{m=0}^{n} \binom{n}{m} x^m y^{n-m}$$

$n=2$: $(x+y)^2 = x^2 + 2xy + y^2$

$n=3$: $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$\vdots$

Okay, now let's discuss dividing objects into multiple groups. We can think of $\binom{n}{m}$ as dividing objects into two groups (the m chosen and the $n-m$ unchosen).

<u>Example:</u>

120 students signed up for a class. The class is divided into 4 sections: S_1, S_2, S_3, S_4, which will have 25, 30, 31, and 34 students, respectively. How many ways are there to divide the students among the four sections?

We can number the students: $1, 2, 3, \ldots, 120$. Then there are

$$\underbrace{\frac{1}{S_1}\;\frac{2}{S_1}\cdots\frac{25}{S_1}}_{\text{25! possible ways to order}}\;\underbrace{\frac{26}{S_2}\cdots\frac{55}{S_2}}_{\text{30! orderings}}\;\frac{56}{S_3}\cdots\frac{86}{S_3}\;\frac{87}{S_4}\cdots\frac{120}{S_4}$$

$$\Rightarrow \frac{120!\text{ permutations}}{25!\,30!\,31!\,34!\text{ possible orderings}}$$

Theorem:

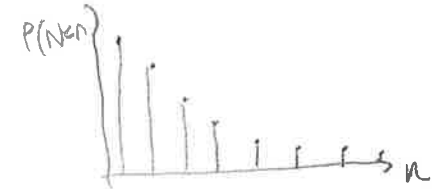
The number of ways to group n objects into m groups of size $n_1, \dots, n_m$ with $n_1 + n_2 + \cdots + n_m = n$, is

$$\frac{n!}{n_1!\, n_2!\, n_3! \cdots n_m!} = \binom{n}{n_1, n_2, n_3, \dots, n_m}$$

Section 2.2: Binomial and multinomial distributions

Distributions we've discussed thus far:

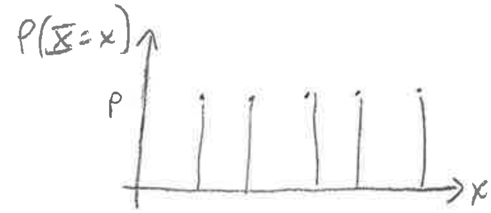
Geometric: $P(N=n) = (1-p)^{n-1} p$



Ex:
Defective parts: 2 outcomes, test until one outcome occurs
(independent)

Uniform: $P(X=x) = p$ (all outcomes equally likely)

Ex:
Rolling a fair die; flipping a coin once.



New distribution: binomial.

Suppose we are testing diodes in batches (so examining n ind. diodes at a time, each diode having a probability p of success). Let X be the number of items that are successful. Let 0 indicate failure, 1 indicate success. Then the outcomes are

$X=1$: $\overbrace{000\cdots0}^{n \text{ times}}$

$X=2$: $100\cdots0$ $\quad 010\cdots0$ $\quad 001\cdots0$ $\quad \cdots$ $\quad 000\cdots01$

$X=3$: $110\cdots0$ $\quad \cdots$

So

$$P(X=x) = \underbrace{p^x}_{x \text{ successful}} \overbrace{(1-p)^{n-x}}^{n-x \text{ fail}} \underbrace{C_{n,x}}_{\text{all possible combinations of orders}} = p^x (1-p)^{n-x} \binom{n}{x}$$

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Question:

What would change if we called p the probability of failure?

Nothing - because of symmetry of binomial coefficients
(numerically)

Definition:

The binomial(n, p) distribution of a discrete r.v. is defined to be

$$P(X=x) = p^x(1-p)^{n-x}\binom{n}{x},$$

where n is the number of trials in a batch and p is the probability of each trial's outcome.

Note:

1. Sometimes this is also referred to as the binomial dist. with parameters n and p.
2. A r.v. with a binomial dist. is said to be a binomial r.v. with parameters n and p.

Theorem:

The binomial(n, p) distribution has mean np.

Let p be the prob. of success.

Proof:

Let $X_i = 1$ if the i^{th} trial is a success and 0 otherwise. Then

$$S_n = X_1 + X_2 + \cdots + X_n$$

is the number of successes in n trials, so

$$ES_n = E(X_1 + X_2 + \cdots + X_n) = EX_1 + EX_2 + \cdots + EX_n$$
$$= p + p + \cdots + p = np$$

Alternatively, we can go from the definition of the expectation

$$ES_n = \sum_{m=1}^{n} m P(S_n = m) = \sum_{m=1}^{n} m\, p^m (1-p)^{n-m} \binom{n}{m}$$

$$= \sum_{m=1}^{n} m \frac{n!}{m!(n-m)!} p^m (1-p)^{n-m}$$

$$= np \underbrace{\sum_{m=1}^{n} m \frac{(n-1)!}{m!(n-m)!} p^{m-1} (1-p)^{n-m}}$$

$= np$ (*) = expectation of total prob. of binom dist.

Total probability:

$$1 = P(S_n \le n) = P(S_n=0) + P(S_n=1) + P(S_n=2) + \dots + P(S_n=n)$$

$$= p(1-p)^{n-1}\binom{n}{1} + \binom{n}{2}p^2(1-p)^{n-2} + \dots + \binom{n}{n}p^n(1-p)^{n-n}$$

$$= p(1-p)^{n-1}\left(\frac{n!}{1!(n-1)!}\right) + \frac{n!}{2!(n-2)!}p^2(1-p)^{n-2} + \dots + \frac{n!}{n!(n-n)!}p^n(1-p)^0$$

$$= np\Big[\frac{(n-1)!}{1!(n-1)!}p^0(1-p)^{n-1} + \frac{(n-1)!}{2!(n-1)!}p(1-p)^{n-2} + \dots + \frac{(n-1)!}{n!(n-n)!}p^{n-1}(1-p)^0$$

$\rightarrow ES_n =$ (*)

Note:

The binomial distribution is typically singly-peaked:

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Example:

Consider an experiment where a fair coin is tossed 10 times.
Let X be the # of heads obtained. Then

$$\Omega = \{0, 1, 2, \dots, 10\}$$

$\forall x$, $P(X=x)$ is the sum of all probabilities of the outcomes in the event $X=x$. Each outcome has the same probability: $1/2^{10}$.

We know $X = x$ iff exactly x of the tosses are H, so the number of outcomes with $X=x$ is the number of subsets of size x that can be chosen from 10 tosses: $\binom{10}{x}$. So

$$P(X=x) = \binom{10}{x}\frac{1}{2^{10}}$$

which is binomial$(10, \frac{1}{2})$.

Question:

What about more than two possible outcomes?

Example:

Consider a die with 1 painted on 3 sides, 2 on two sides, and 3 on one side. If rolling 10x, what is prob of getting 5 1's, 3 2's, and 2 3's?

$$\underbrace{\frac{10!}{5!\,3!\,2!}}_{\text{\# ways to pick 5 rolls for 1's, 3 rolls for 2's, two rolls for 3's}}\left(\frac{1}{2}\right)^5\left(\frac{1}{3}\right)^3\left(\frac{1}{6}\right)^2$$

Definition:

Multinomial dist: $P(X_1=x_1, X_2=x_2, \dots, X_k=x_k) = \frac{n!}{x_1!\,x_2!\cdots x_k!}\,p_1^{x_1}\,p_2^{x_2}\cdots p_k^{x_k}$

Section 2.3: Poisson approximation to the binomial

The Poisson distribution is another of the most common distributions (along with the uniform and the binomial distributions). It can be viewed as arising from the binomial distribution.

Example:

Suppose we are estimating phone calls to a police station in a large city, such as the probability that more than 10 phone calls occur in a 5-minute time interval.

We must assume an average rate of λ calls/minute, so in a 5-minute interval we expect about 5λ calls. We must also assume that the number of calls in non-overlapping time intervals is independent.

Model the number of calls in an interval of length t as a binomial random variable, X

Question:
why can we do this? (How does this align with our discussion of binomial r.v.'s last time?)

↳ Modeling 'batches' of n occurrances in each time interval

Then for each interval of length t, there are λt occurances on avg, and because we're modeling as a binomial r.v., the expected value is also np. Hence,

$$\lambda t = np \rightarrow p = \frac{\lambda t}{n}$$

Let's fix the time interval length: $t=1$.

We can now derive the distribution of possible outcomes.

$$P(X=0) = \binom{n}{0} p^0 (1-p)^{n-0} = (1-p)^n = \left(1-\frac{\lambda}{n}\right)^n$$

0 calls in 1 interval

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For large n,

$$\left(1-\frac{\lambda}{n}\right)^n \sim e^{-\lambda}$$

Recall:
$$\boxed{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda} \text{ by definition}}$$

__Question:__
what is $P(X=1)$?

$$P(X=1) = \binom{n}{1} p^1 (1-p)^{n-1} = \frac{n!}{1!(n-1)!}\left(\frac{\lambda}{n}\right)^1\left(1-\frac{\lambda}{n}\right)^{n-1}$$

$$= \frac{\frac{\cancel{n}\lambda}{\cancel{n}}\left(1-\frac{\lambda}{n}\right)^n}{\underbrace{\left(1-\frac{\lambda}{n}\right)^k}_{\to 1}}$$

n large $\Rightarrow$ $P(X=1) \approx \lambda e^{-\lambda}$

In general,

$$P(X=k) \approx \frac{\lambda^k}{k!}e^{-\lambda}$$

(This is the Poisson distribution)

__Definition:__
A r.v. X has a __Poisson distribution__ with parameter λ, or Poisson(λ), if

$$P(X=k) = \frac{\lambda^k}{k!}e^{-\lambda}, \quad k=0,1,2,\ldots$$

__Question:__
what is $P(X \le k)$? Or rather the sum of the possible probabilities?
1, as always! $\left(P(X\le k) = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}e^{-\lambda} = e^{-\lambda}\underbrace{\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}}_{\text{power series for } e^{\lambda}} = e^{-\lambda}e^{\lambda}\right.$

Theorem:

The Poisson distribution has mean λ and variance λ

Proof of $E\bar{X}=\lambda$:

$$E\bar{X} = \sum_{k=0}^{\infty} k e^{-\lambda} \frac{\lambda^k}{k!} = \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^k}{(k-1)!} = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda} e^{\lambda} = \lambda$$

Note:

Again, $E(\bar{X}(\bar{X}-1)) = \sum_{k=2}^{\infty} k(k-1) e^{-\lambda} \frac{\lambda^k}{k!}$

General Poisson:

In general, we may not want to assume that each interval has the same probability of events occuring (e.g., car crashes happen more frequently at different hours of the day). We thus have a generalization of the Poisson distribution that can bound

Theorem:

Consider independent events A_i, $i=1,2,\dots,n$, with probabilities $p_i = P(A_i)$. Let N be a r.v. for the number of events that occur, let $\lambda = p_1 + p_2 + \dots + p_n$, and let Z have a Poisson distribution with parameter λ. Then, $\forall B \subset \mathbb{Z}$,

$$|P(N \in B) - P(Z \in B)| \leq \sum_{i=1}^{n} p_i^2$$

Note:

$$\sum_{i=1}^{n} p_i^2 \leq p_1 p_1 + p_2 p_2 + \dots + p_n p_n \leq (\max_i p_i) \sum_{i=1}^{n} p_i = \lambda \max_i (p_i)$$

So if all the p_i's are small, the distribution for N is close to a Poisson dist. with parameter λ.

Section 2.4: ~~Card games and other~~ urn problems

More realistically, given time

Now for some fun counting problems! Urn problems are classic problems in probability, and we can re-imagine many 'non-urn' counting problems in this style to solve them.

Example:

Pick 4 balls out of an urn with 12 red balls and 8 black balls. What is the probability of the event

"B = we get two balls of each color"

Question:

what are the categories of things we need to count here?

① # ways to pick 4 balls out of 20 ($\Rightarrow |\Omega|$)

size of sample space

② # ways to pick 2 red and 2 black?

① $\binom{20}{4} = \frac{20 \cdot 19 \cdot 18 \cdot 17}{4!} = 4,845 = |\Omega|$

② $|B| = \underbrace{\binom{12}{2}}_{\substack{\text{\# ways}\\ \text{to choose}\\ \text{2 red}}} \underbrace{\binom{8}{2}}_{\substack{\text{\# ways}\\ \text{to choose}\\ \text{2 black}}} = \frac{12 \cdot 11}{2!} \cdot \frac{8 \cdot 7}{2!} = 1,848$

$\Rightarrow P(B) = \frac{|B|}{|\Omega|} = \frac{1848}{4845} \approx 0.38$

(I'd suggest reading the section for many other examples)

(If time permits)

Example: Capture-recapture experiments

An ecology student captures $k=60$ beetles at a pond, marks each with a dot of paint, and releases them. A few days later they capture $r=50$ beetles, finding $m=12$ marked and $r-m=38$ unmarked beetles. What is the best guess for the size of the total population?

Question: If N beetles in pond, prob. of getting m marked and $r-m$ unmarked in a sample of r is

$$p_N = \frac{\binom{k}{m}\binom{N-k}{r-m}}{\binom{N}{r}}$$

Note: ① To estimate the population, we pick N to maximize p_N. This is called the maximum likelihood estimate.

② we need to note that

$$\binom{j-1}{i} = \frac{(j-1)!}{(j-i-1)!\,i!} \Rightarrow \binom{j}{i} = \frac{j}{j-i}\binom{j-1}{i}$$

We have that

$$\frac{p_N}{p_{N-1}} \geq 1 \iff p_N = p_{N-1}\left(\frac{N-k}{N-k-(r-m)}\right)\left(\frac{N-r}{N}\right) \geq p_{N-1}$$

prob higher for pop. N than $N-1$

$$\iff N^2 - kN - rN + kr \geq N^2 - kN - rN + mN$$

$$\iff N \leq \frac{kr}{m}$$

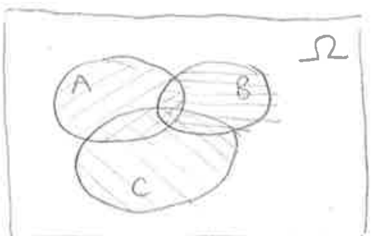
For beetles, $\frac{kr}{m} = 250$, so probability is maximized when $N=250$

Section 2.5: Properties of Unions
(aka Inclusion-Exclusion Principle)

In Section 1.1, we learned that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Since then, we've talked about how we can determine similar formulas by being careful about where we're double-counting.

For example:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$



$\equiv$ = $P(A)$ $\#$ = $P(A \cap B)$

$\backslash\backslash$ = $P(C)$ $\not\equiv$ = $P(B \cap C)$

$\equiv$ = $P(B)$ $\times$ = $P(A \cap C)$

Here's a generalization for the union of n events:

<u>Theorem</u> (Inclusion-Exclusion)

Let $A_1, \ldots, A_n$ be events. Then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k)$$

$$- + \cdots - (-1)^{n+1} P(A_1 \cap \cdots \cap A_n)$$

I.e., we take all possible intersections of $1, 2, \ldots, n$ events. To account for double-counting, the signs alternate.

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Proof:

Let an outcome be in k events. That outcome is counted

$+ \quad k$ times in $\sum_{i=1}^{n} P(A_i)$

$- \quad \binom{k}{2}$ times in $\sum_{i<j} P(A_i \cap A_j)$

$+ \quad \binom{k}{3}$ times in $\sum_{i<j<\ell} P(A_i \cap A_j \cap A_\ell)$

$\vdots \qquad \vdots$

$(-1)^{k+1} \quad \binom{k}{k}$ times in $P(A_1 \cap A_2 \cap \cdots \cap A_k)$

Then we have

$$\underbrace{\binom{k}{1}}_{k} - \binom{k}{2} + \binom{k}{3} - + \cdots + (-1)^{k+1} \underbrace{\binom{k}{k}}_{=1}$$

This sums to 1, by the binomial theorem: notice

$$(a+b)^k = a^k + \binom{k}{1} \underbrace{a^{k-1} b}_{-} + \binom{k}{2} \underbrace{a^{k-2} b^2}_{+} + \cdots + b^k \binom{k}{k}$$

Let $a=1, b=-1$: Then

$$0^k = 1^k - \binom{k}{1} + \binom{k}{2} - \binom{k}{3} + - \cdots + (-1)^{k+1} \binom{k}{k}$$

$$\Rightarrow \quad 1 = \binom{k}{1} - \binom{k}{2} + \binom{k}{3} - + \cdots - (-1)^{k+1} \binom{k}{k}$$

This says each outcome is counted exactly once

~~Example:~~

~~An urn contains 30 red, 20 green, and 10 yellow balls. Draw 2 without replacement. What is the probability the sample contains 1 red or 1 yellow?~~

Example:

Suppose n people leave their hats in a cloakroom, but the attendant mixes the hats up, so each person leaves with a random hat. (Assume all $n!$ assignments of hats are equally likely.) What is the probability that no one gets their own hat?

Let

A_i = "person i gets their own hat", $i = 1, \ldots, n$

Then we want

$$P\left(\bigcap_{i=1}^{n} A_i^c\right) = P(A_1^c \cap A_2^c \cap \cdots \cap A_3^c)$$

$$= P((A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n)^c) \quad \text{by de Morgan's Law}$$

$$= 1 - P(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n)$$

$$= 1 - \left[\sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + - \cdots + (-1)^{k+1} P(A_1 \cap \cdots \cap A_n)\right]$$

$= (*)$ by Inclusion-Exclusion

so the probability that k people get their own hats is:

$$P(A_1 \cap A_2 \cap \cdots \cap A_k) = \frac{(n-k)!}{n!}$$

$(n-k)!$ ← $n-k$ ways hats can be miss-assigned

$n!$ ← n ways to arrange the hats

(there is $\frac{1}{n!}$ way each hat can be assigned correctly)

Then

$$\sum_{i_1 < i_2 < \cdots < i_k} P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = \binom{n}{k}\frac{(n-k)!}{n!} = \frac{n!}{(n-k!)k!}\left(\frac{(n-k)!}{n!}\right) = \frac{1}{k!}$$

?

Then, going back to (*),

$$P\left(\bigcap_{i=1}^{n} A_i^c\right) = 1 - \left[1 - \frac{1}{2!} + \frac{1}{3!} - + \cdots + (-1)^{n+1}\frac{1}{n!}\right]$$

$$= 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}$$

$$= \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

Notice:

This is the beginning of the series representation of e^x, evaluated at $x=-1$. So, as $n \to \infty$, $P\left(\bigcap_{i=1}^{n} A_i^c\right) \to \frac{1}{e}$

Example:

Pick 7 cards out of a deck of 52. What is probability of three of a kind (e.g. 3 Kings or 3 Aces)

Let A_i, $i \in [1,13]$ be the event we have 3 cards of type i. At most two of these events can occur (picking 7 cards), so

$$P\left(\bigcup_{i=1}^{13} A_i\right) = 13P(A_1) - \binom{13}{2}P(A_1 \cap A_2)$$

for some kind, you have 3

$$= \frac{13\binom{4}{3}\binom{48}{4}}{\binom{52}{7}} - \frac{\binom{13}{2}\binom{4}{3}\binom{4}{3}\binom{44}{1}}{\binom{52}{7}}$$

have 4 from the remaining 48

$$= \frac{13(778{,}320) - 78(704)}{\binom{52}{7}}$$

$$\approx 0.075$$

Bonferroni inequalities

If you stop the inclusion-exclusion formula with a + term, you get an upper bound; if you stop with a − term you get a lower bound:

$$P\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} P(A_i)$$

$$\ge \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j)$$

$$\le \quad \text{''} \qquad\qquad \text{''} \qquad + \sum_{i<j<k} P(A_i \cap A_j \cap A_k)$$

Section 2.6: Blackjack

Let's talk about how to win at blackjack.

The rules of the game.

Dealer's rule: draw if total ≤ 16
otherwise, stop

Question 1:
What is probability the dealer's ending total is k when their initial total value is j?

Definition:
A soft hand has a total of j including an ace that is being counted as 11.

$b(j,k)$ = prob. dealer's ending total is k when has a total of j with a soft hand

A hard hand has an ace counted as 1

$a(j,k) = \ldots$

Note:
$a(j,j) = b(j,j) = 1 \quad , \quad j \geq 17$

If $j \in [11, 16]$, a new ace counts as 1

$$p_i = \begin{cases} \frac{1}{13} & , i \in [1,9] \\ \frac{4}{13} & , i = 10 \end{cases}$$

$$a(j,k) = \underbrace{\left(\frac{1}{13}\right)}_{p_1} a(j+1,k) + \sum_{m=2}^{10} p_m \, a(j+m,k)$$

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Already has an ace as 1

$a(16, 17) = P(\text{Ace}) =$

$b(16, 17) = P(\text{Ace})$

$c(16, 17) =$