Chapter 4: Markov Chains

Def:

A Markov chain is a discrete time random ('stochastic') model that consists of a sequence of random variables (X_n) that

a) Have a discrete set of states:

$$X_n \in \{x_1, x_2, \dots, x_m\}$$

b) Has the 'Markov Property': $P(X_{n+1} = x_j)$ depends only on X_n, the most recent state. The prob. of reaching state x_j when at x_i is the transition probability,

$$\Rightarrow \text{matrix!} \nearrow P_{ij} = p(i,j) = P(X_{n+1} = x_j \mid X_n = x_i)$$

Note:

1. So the sequence of the random variables (X_n) is detesmined by each P_{ij} and the prob dist for the initial value, X_0.
2. Due to the Markov Property, the r.v.'s X_n are not independent

Ex:
Think of a Markov chain as a child playing 'the floor is lava'. They jump from one object to another about a room, with appropriate transition probabilities.

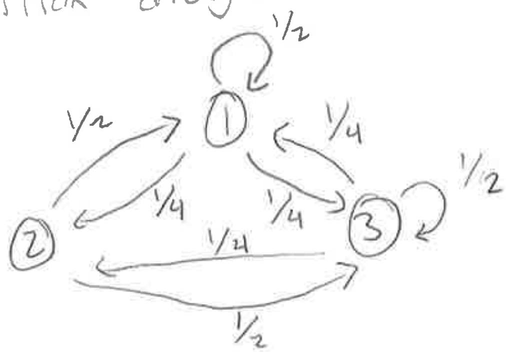
Ex:
Land where it's never nice two days in a row. If today is nice, tomorrow has an equal chance of being rainy or snowy. If snow or rain today, 2x as likely to be same tomorrow.

and equally likely to be diff. from the original state

we can summ. prob. in a transition matrix,

$$\begin{array}{cc} & \text{tomorrow} \\ & \begin{array}{ccc} R & N & S \end{array} \\ \begin{array}{c} 1=R \\ 2=N \\ 3=S \end{array} & P = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix} \\ \text{today} & \end{array}$$

we can also visualize transition probabilities w/ a state transition diagram.



Note:
Any matrix with propy:
1) $p_{ij} \geq 0$
2) $\sum_j p_{ij} = 1$
gives rise to a Markov chain

Question:
If rain today, what is prob. of snow tomorrow?

$$P_{13} = 1/4$$

on 3/7

Example: Wright-Fisher model

Consider N genes with one of two alleles, A or a. Let X_n be # A alleles at time / generation n.

$\boxed{?}$ state space: $\{0, 1, 2, \dots, N\}$

transition probabilities: Binomial! Probability: $\frac{X_n}{N} = \frac{i}{N}$

$$P(X_{n+1} = j \mid X_n = i) = p_{ij} = \binom{N}{j}\left(\frac{i}{N}\right)^j\left(1-\frac{i}{N}\right)^{N-j}$$

$0 \leq i, j \leq N$

i.e., each of N individuals in generation $n+1$ 'pick their parents at random' from pop at time n.

Note:

$$i=0 \Rightarrow P(X_{n+1}=0 \mid X_n=0) = \binom{N}{0}(0)^0(1-0)^{N-0} = 1$$

$$\cancel{i=1 \Rightarrow P(X_{n+1}=1 \mid X_n=1) = \binom{N}{1}(1)^1(1-1)^{N-1}}$$

$$i=N \Rightarrow P(X_{n+1}=N \mid X_n=N) = \binom{N}{N}\left(\frac{N}{N}\right)^N\left(1-\frac{N}{N}\right)^{N-N}$$

$$= 1$$

So if we ever reach these states, we become 'trapped'

Def:

An absorbing state of a Markov chain is one in which, once arrived at, the chain can never leave.

Ex: Two-stage Markov chains

we've talked so far about dist. X_{n+1} depending only on X_n.

We can imagine cases that depend on multiple prior states.

Consider a basketball player who makes a shot w/ prob:

$\frac{1}{2}$ if he missed the last 2 times

$\frac{2}{3}$ if he has hit one of his last two shots

$\frac{3}{4}$ if he has hit both of his last two shots

$\Omega = \{HH, HM, MH, MM\}$

$$P = \begin{array}{c|cccc} & HH & HM & MH & MM \\ \hline HH & \boxed{3/4} & 1/4 & 0 & 0 \\ HM & 0 & 0 & \boxed{2/3} & 1/3 \\ MH & \boxed{2/3} & 1/3 & 0 & 0 \\ MM & 0 & 0 & \boxed{1/2} & 1/2 \end{array}$$

$X_n \quad X_{n-1} \quad X_n$

$$P(MH \mid HM) = \frac{2}{3}$$

X_{n+1}

Section 4.2: Multistep transition probabilities

Question:
If rain today, what is prob of snow the day after tomorrow?

$$p_{11}p_{13} + p_{12}p_{23} + p_{13}p_{33} = p_{13}^2$$

$$= \frac{1}{2}\left(\frac{1}{4}\right) + \frac{1}{4}\left(\frac{1}{2}\right) + \frac{1}{4}\left(\frac{1}{2}\right)$$

$$= \frac{3}{8}$$

Note:

$p_{13}^2 = \langle p_{11}, p_{12}, p_{13} \rangle \cdot \langle p_{13}\ p_{23}\ p_{33} \rangle$

↖ dot product of 1st row and 3rd column

In general, the probability of transitioning from and state i to state j in 2 steps is given by

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} = \underline{P}^2$$

In general,

Theorem:
The m-step transition probability

$$p^m(i,j) = P(X_{n+m} = j \mid X_n = i)$$

is the m^{th} power of the transition matrix

This result is due to the Chapman-Kolmogorov equation

$$p^{m+n}(i,j) = \sum_k p^m(i,k)\,p^n(k,j) \qquad (*)$$

Note:

This type of question implicitly includes all possible intermediate steps: when considering an n-step transition probability: e.g., in a 3-state Markov chain,

$$p_{13}^3 = \sum_{j=1} \sum_{k=1} p(1,j)p(j,k)p(k,3)$$

whereas prob going from 1 to 2 to 3 is

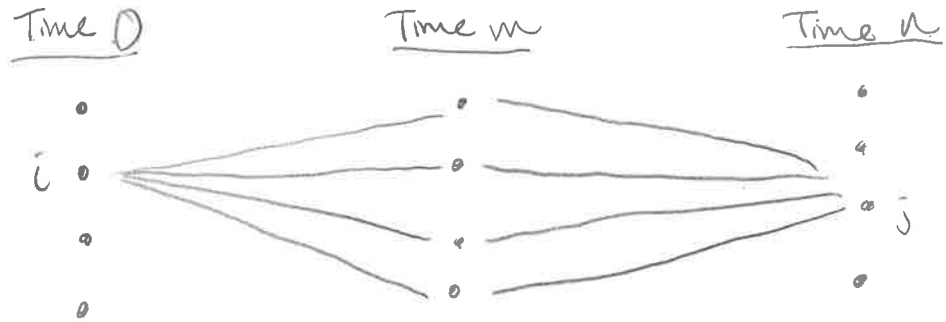
$$p(1,2)p(2,3)$$

So in the worksheet last time, HTHH occurs with probability

$$\underbrace{\frac{1}{2}\left(\frac{1}{2}\right)}_{\text{Initial state: } p(HT)} p(HT,TH)\,p(TH,HH) = \frac{1}{16}$$

Discussion:

The Chapman-Kolmogorov equation says that, to go from i to j in m+n steps, we go between i and k in some m steps, and then between k and j in some n steps:



The Markov Property implies that these two 'steps' are independent

$m, n > 1$, possibly, so steps of steps

We can prove (*) by considering the definition of an m-step transition probability,

$$p^{m+n}(i,j) = P(X_{m+n} = j \mid X_0 = i)$$

$$= \sum_k P(\{X_{m+n} = j\} \cap \{X_m = k\} \mid X_0 = i)$$

Law of Total Prob. for conditional probabilities: (worksheet)

$$= \vdots = \Big\} \text{worksheet}$$

$$= \sum_k p^m(i,k)\, p^n(k,j)$$

(Not quite correct as stated in book - what's the typo?)

Section 4.6: Absorbing chains

(insert stuff from 3/5)

So the example from the worksheet last time about the Master's program and the Gambler's Ruin both had absorbing states, while the coin flip problem didn't.

One question we can ask with absorbing chains is, when you consider the absorbing states, how long (in some sense) might it take the chain to enter these states?

Example:

An office computer is in one of three states: working (W), being repaired (R), or scrapped (S). If the computer is working 1 day, the prob working the next is 0.995, and prob needing repair is 0.005. If being repaired, prob working next day is 0.05, + prob scrapped is 0.05.

What is the avg # working days until a comp is scrapped?

The book immediately drops the absorbing states and considers only the transient states:

$$Q = \begin{array}{c} \\ W \\ R \end{array} \begin{array}{c} \begin{array}{cc} W & R \end{array} \\ \begin{bmatrix} 0.995 & 0.005 \\ 0.90 & 0.05 \end{bmatrix} \end{array}$$

As $n \to \infty$, $Q^n \to \underset{\sim}{0}$ (matrix)

Notice, $Q^n(w,w)$ gives the prob. a computer is working on day n.
Let $Y_n = 1$ if the comp is working on day n, 0 otherwise.

Then $E\,Y_n = Q^n(w,w)$. So the total # days the comp is working is $Y = \sum_{n=0}^{\infty} Y_n$, and

$$EY = E\sum_{n=0}^{\infty} Y_n = \sum_{n=0}^{\infty} E\,Y_n = \sum_{n=0}^{\infty} Q^n(w,w)$$

Claim:

$$\sum_{n=0}^{\infty} Q^n = N \equiv (I-Q)^{-1}$$

Proof: $(I-Q)\vec{x} = 0$, so let $\vec{x} = Q\vec{x}$. Then $\vec{x} = Q^n\vec{x}$. Since $Q^n \to 0$, $Q^n\vec{x} \to 0$, so $\vec{x} = 0$. Hence, $(I-Q)^{-1} = N$ exists.

Note that

$$(I-Q)(I+Q+Q^2+\cdots+Q^n) = I - Q^{n+1}$$

$$\Rightarrow \underbrace{N(I-Q)}_{=I}(I+Q+Q^2+\cdots+Q^n) = N(I-Q^{n+1})$$

$n\to\infty \Rightarrow$

$$N = I + Q + Q^2 + \cdots = \sum_{n=0}^{\infty} Q^n \qquad \square$$

In other words, the expected number of working days is

$$EY = \sum_{n=0}^{\infty} Q^n(w,w) = (I-Q)^{-1}(w,w)$$

so,

$$(I-Q)^{-1} = \begin{bmatrix} 0.005 & 0.005 \\ 0.90 & 0.95 \end{bmatrix}^{-1} = \begin{bmatrix} 3800 & 20 \\ 3600 & 20 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Definition: For an absorbing markov chain P, the matrix $N = (I-Q)^{-1}$ is the fundamental matrix for P. The entry $N(i,j)$ gives the expected number of times the process is in the transient state s_j if it is started in the transient state s_i.

Another question we can ask w/ absorbing chains is, what is prob of entering an absorbing state?

EX: Tennis (1st player to win 4 pts, unless 4-3/cont until ahead by 2)
Suppose one player is trying to get ahead 2 pts to win, + server indep. wins a point w/ prob. 0.6.

We need one score for the Markov chain: let the state be the difference of scores

$\Omega = \{2, 1, 0, -1, -2\}$

$2 \to$ win for server
$-2 \to$ win for opponent

$$P = \begin{array}{c} \\ 2 \\ 1 \\ 0 \\ -1 \\ -2 \end{array} \begin{array}{c} \begin{array}{ccccc} 2 & 1 & 0 & -1 & -2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.6 & 0 & 0.4 & 0 & 0 \\ 0 & 0.6 & 0 & 0.4 & 0 \\ 0 & 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 0 & & 1 \end{bmatrix} \end{array}$$

Let $h(x)$ be prob. of the server winning when the score is x. Then

$h(2) = 1$

$h(-2) = 0$

$\Rightarrow h(x) = \sum_y p(x,y)h(y)$

3/7-6

So

$$h(1) = p(1,2)\overbrace{h(2)}^{=1} + p(1,1)h(1) + p(1,0)h(0) + p(1,-1)h(-1)$$

$$+ p(1,-2)\underbrace{h(-2)}_{=0}$$

$$P^2 = \begin{array}{c|ccccc} & 2 & 1 & 0 & -1 & -2 \\ \hline 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0.6 & 0.24 & 0 & 0.16 & 0 \\ 0 & 0.36 & 0 & 0.48 & 0 & 0.16 \\ -1 & 0 & 0.36 & 0 & 0.24 & 0.4 \\ -2 & 0 & 0 & 0 & 0 & 1 \end{array}$$

$h(0)$

similar

$h(-1)$

Consider P^2:

$$h(0) = p(0,2)h(2) + p(0,1)\overbrace{h(1)}^{0} + p(0,0)h(0)$$

$$+ p(0,-1)\underbrace{h(-1)}_{0} + p(0,-2)\underbrace{h(-2)}_{0}$$

$$= 0.36 + 0.48\,h(0)$$

same unknown!

$$\Rightarrow h(0) = \frac{0.36}{0.52} \approx 0.6923$$

Then $h(1) = 0.6 + 0.4\,h(0) \approx 0.8769$

$h(-1) = 0.6\,h(0) \approx 0.4154$

Section 4.3: Stationary distributions

Last time, we discussed finding the eventual probability of entering a state, given an initial state or set of states. That begs the question, if we have a probabilistic representation of the possible states at time n, what happens to that distribution as $n \to \infty$?

i.e., what is $Pr\{X = j\}$ for large n?

Ex:

Consider $p = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 7/10 & 3/10 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$X_0 = 1 \Rightarrow P(X_1 = 1) = \frac{1}{3}$

$P(X_2 = 1) = p_{11} P(X_1 = 1) + p_{21} P(X_1 = 2) + p_{31} P(X_1 = 3)$

$= \frac{1}{3}\left(\frac{1}{3}\right) + \frac{7}{10} \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} \approx 0.678$

In general,

$$P(X_{n+1} = j) = \sum_i p_{ij} P(X_n = i)$$

Let $\pi_n(i) = P(X_n = i)$. Then

$$\pi_{n+1}(j) = \sum_i p_{ij} \pi_n(i)$$

$$\Rightarrow (\pi_{n+1}(1), \pi_{n+1}(2), \ldots, \pi_{n+1}(k)) = (\pi_n(1), \pi_n(2), \ldots, \pi_n(\ell)) \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1k} \\ p_{21} & p_{22} & \cdots & \\ \vdots & & & \\ p_{\ell 1} & \cdots & & p_{\ell k} \end{bmatrix}$$

i.e.,

$$\vec{\pi}_{n+1} = \vec{\pi}_n P$$

If we iterate this map, as $n \to \infty$, we say that the limiting distribution (if it exists), is called the steady-state distribution,

$$\vec{\pi}_n \to \vec{\pi}.$$

Note:
when it exists, we can calculate the steady-state distribution $\vec{x}$ of the state distribution $\vec{x}_n$ as $n \to \infty$ by letting

$$\vec{x}_n \to \vec{x} \quad \text{and} \quad \vec{x}_{n+1} \to \vec{x},$$

so

$$\vec{x} = \vec{x} P \quad \to \quad \vec{x}(I - P) = \vec{0} \quad , \quad \sum_{i=1}^{m} \vec{x}(i) = 1$$

Return to example:

$$P = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 7/10 & 3/10 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \pi_0 = ?$$

we had $\pi_0 = (1,0,0)$

$$\to \pi_1 = (1,0,0)P = (1/3, 1/3, 1/3)$$

Quick question:
How would you show that $\vec{\pi}_{n+1}$ is a probability distribution, for a general prior distribution $\vec{\pi}_n$?

$$\sum_j \pi_{n+1}(j) = \sum_i \sum_j p_{ij} \vec{\pi}_n(i) = \sum_i \vec{\pi}_n(i) \underbrace{\sum_j p_{ij}}_{=1} = 1$$

We can calculate the steady, state distribution by finding $\vec{\pi}$ s.t.

$$\vec{\pi} = \vec{\pi} P$$

$$(\pi_1, \pi_2, \pi_3) = (\pi_1, \pi_2, \pi_3) \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 7/10 & 3/10 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \pi_1 = \frac{1}{3}\pi_1 + \frac{7}{10}\pi_2 + \pi_3$$

solve for π_1, π_2, π_3

$$\pi_2 = \frac{1}{3}\pi_1 + \frac{3}{10}\pi_2 + 0$$

$$\pi_3 = \frac{1}{3}\pi_1$$

OR solve it in matrix form. As it stands, we would get

$$(\pi_1, \pi_2, \pi_3) A = 0$$

$\leftarrow$ i.e. find kernel of A.

An easier way would be to note that one of the equations is redundant: summing both sides gives

$$\pi_1 + \pi_2 + \pi_3 = \pi_1 + \pi_2 + \pi_3$$

Replacing the 3rd equation by $\pi_1 + \pi_2 + \pi_3 = 1$, we can write this system as

$$0 = -\frac{2}{3}\pi_1 + \frac{7}{10}\pi_2 + \pi_3$$

$$0 = \frac{1}{3}\pi_1 - \frac{7}{10}\pi_2$$

$$1 = \pi_1 + \pi_2 + \pi_3$$

$$\rightarrow (0,0,1) = (\pi_1, \pi_2, \pi_3) \overbrace{\begin{bmatrix} -\frac{2}{3} & 1/3 & 1 \\ 7/10 & -7/10 & 1 \\ 1 & 0 & 1 \end{bmatrix}}^{A} \rightarrow (0,0,1)A^{-1} = \pi$$

Section 4.4

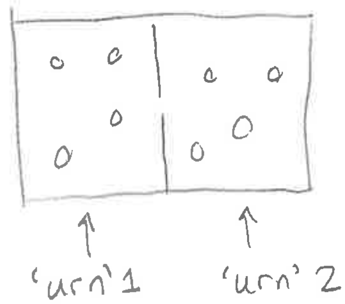
3/12-4

Motivating question:

How do we ensure that a stationary distribution exists?

Example: Ehrenfest chain

The Ehrenfest chain model is a model for the exchange of gas molecules between two chambers



pick 1 of N balls at random & move to other urn.

Let X_n be the # of balls in urn 1 after the n^{th} draw.

Suppose $N=2$. Then the transition matrix is

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 0 \\ 1 & 1/2 & 0 & 1/2 \\ 2 & 0 & 1 & 0 \end{array}$$

possible to go from each state to each other state

Example:

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \to P^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\to P^n = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & n \text{ even} \\ P, & n \text{ odd} \end{cases}$$

3/12-5

Characteristics of Markov chains:

Discrete continues

1. A Markov chain is ergodic, or irreducible, if it is possible to go from every state to every state (not nec. in one step); ie, $p^m(i,j) > 0$ for some $m \geq 1$

   ↳ If it is not possible, we can say the chain is reducible

2. A Markov chain is regular if some power of the tran. matrix has only positive entries. I.e. $\exists n$ s.t. (↑ non-zero) st. it is possible to go from any state to any other state in n steps

Note: regular $\Rightarrow$ ergodic

EX:
EX2 + EX 1 ~~is~~ are ergodic, but not regular chains

3. A state i of $p^n(i,j)$ is said to be aperiodic if

$$GCD(J_i) = 1$$

More next time

where

$$J_i = \{n \geq 1 \text{ s.t. } p^n(i,i) > 0\}$$

The $GCD(J_i)$ is called the period of state i

Markov Chain Convergence Theorem:

If p is irreducible and has an aperiodic state, then $\exists$ a unique stationary distribution $\vec{\pi}$, and $\forall$ i and j

$$p^n(i,j) \rightarrow \vec{\pi}(j) \quad \text{as} \quad n \rightarrow \infty$$

Last time:

Definition:

A state i of $p^n(i,j)$ is said to be aperiodic if

$$GCD(J_i) = 1$$

where

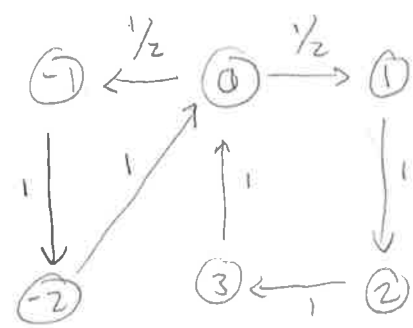
$$J_i = \{n \geq 1 \text{ s.t. } p^n(i,i) > 0\}$$

The $GCD(J_i)$ is called the period of state i

Example:

Consider

$$p = \begin{array}{c} \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} -2 & -1 & 0 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cccccc} - & - & 1 & - & - & - \\ 1 & - & - & - & - & - \\ \cancel{0} & 1/2 & - & 1/2 & - & - \\ - & - & - & - & 1 & - \\ - & - & - & - & - & 1 \\ - & - & 1 & - & - & - \end{array}\right] \end{array}$$



What is J_0?

$$J_0 = \{n \geq 1 \mid \underbrace{p^n(0,0) > 0}\}$$

It's possible to go from 0 to 0 in n steps

$= \{3, 4, \dots\}$

directly $\Rightarrow$ { multiples of 3 and 4 + closed under addition $\Rightarrow$ 3 consecutive integers $\Rightarrow$ all others

$= \{3, 4, 7, 6, 8\} \cup \mathbb{Z}_9$

Then $GCD(J_0) = 1$

in general, aperiodic states have the property that J_i contains all integers beyond some point

Convergence Thm:

If p is irreducible + has an aperiodic state, then $\exists$ unique stat dist π, s.t. $\forall i, j$,

$$p^n(i,j) \to \pi(j) \quad \text{as } n \to \infty$$

Corrollary:

If $\exists n$ s.t. $p^n(i,j) > 0 \ \forall i,j$, then $\exists$ unique state dist π and

$$p^n(i,j) \to \pi(j) \quad \text{as } n \to \infty$$

Proof:

In this case, p is irreducible (possible to go (i,j) in n steps. All states aperidic since $p^{n+1}(i,j) > 0$, so $n, n+1 \in J_i$, so $GCD(J_i) = 1$

$\square$

Question:

which of the models we've discussed can be shown convergent using the convergence theorem?

Ehrenfest chain → no, allstates have period 2

Tennis → no; absorbing chains are not

Computer repair → irreducible

wright-fisher → yes; with mutation:

suppose an A drawn becomes an a in the next gen. w/ prob. u, while an a drawn becomes w/ prob. v. The prob. A is produced by a given draw is

$$\rho_i = \underbrace{\frac{i}{N}(1-u)}_{} + \underbrace{\frac{N-i}{N} v}_{}$$

we can get A by: drawing an A and not having a mutation OR drawing a and having a mutation

Note: Do we consider A drawn & mutating?

↳ No; only considering mutations of 'drawn' alleles

For the case with mutations, the trans. prob. still has the form

$$p(i,j) = \binom{N}{j}(\rho_i)^j (1-\rho_i)^{N-j}$$

b/c draws indep.

Converges by conv. theorem? → HW

Section 4.5: Gambler's Ruin

A very famous problem. Appeared on worksheet last week

Consider a gambling game in which on any turn you win \$1 w/ prob $p=0.4$ or lose \$1 w/ prob $1-p=0.6$. Suppose you quit when your fortune reaches \$N, and you have to if you reach \$0

Questions:

what is the transition matrix?

Is it irreducible and aperiodic?

$\to$ no - absorbing $\Rightarrow$ reducible

So the MCCT doesn't apply. But what are possible stat. dist.'s?

$$\underset{\$N}{(0, \dots, 1)} \quad \text{or} \quad \underset{\$0}{(1, 0, 0, \dots, 0)}$$

<u>Question:</u>

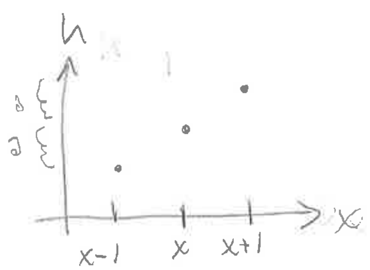
What is the prob. the gambler avoids ruin?

We can think of it as a coin flip. e.g., ~~Suppose~~ the

x(gambler has \$10, and will stop at \$25 = N.

Consider any x: WTF $h(x)$: prob of reaching N before 0.

$$h(0) = 0$$
$$h(N) = 1$$
$$0 < x < N: \quad h(x) = \tfrac{1}{2} h(x-1) + \tfrac{1}{2} h(x+1)$$



$$\Rightarrow 2h(x) = h(x-1) + h(x+1)$$
$$\to h(x) - h(x-1) = h(x+1) - h(x)$$
$$\Rightarrow h(x) \text{ is linear}$$

$h(0)=0 \to m = \frac{1}{N}$
$h(N)=1$

$$\Rightarrow h(x) = \frac{x}{N}$$

So if gambler starts at \$10, prob is $\frac{10}{25}$.