Chapter 5: Continuous distributions

5.1: Density Functions

Now let's more into the realm of continuous random variables. With discrete r.v.'s, we looked at the distribution of possible outcomes by enumerating it as a list:

x	0	1	2	3
$P(X=x)$	$1/6$	$1/2$	$1/6$	$1/6$

There are lots of phenomena that make more sense to be modeled as continuous r.v.'s: e.g.,

- height/weight of a random person in a group

physical variables
- temperature
- voltage

Let's consider how to think about the probability of outcomes of a cont. r.v.

Def:
Let X be a cont. real-valued r.v.. A density function for X is a real valued function, which satisfies

$$P(a \le X \le b) = \int_a^b f(x)\,dx$$

In other words, the prob. that $X \in [a,b]$ is given by the area under $f(x)$ from a to b.

Does not need to be closed

Note:

1. By this definition,

$$P(X=c)=\int_c^c f(x)\,dx = 0 \qquad \forall c \in \mathbb{R}$$

2. $P(x)$ is also called the probability density function, or pdf

Question:

Which functions $f(x)$ qualify as density functions?

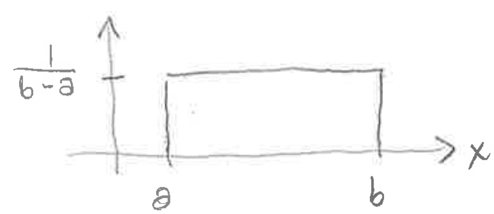
We want our probabilities to be non-negative, and to 'sum to one': $P(-\infty < X < \infty) = 1$. Therefore, the conditions we must place on $f(x)$ are:

1. $f(x) \geq 0 \quad \forall x \in \mathbb{R}$
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

In practice, this is how we check whether a function is a density function

Example 1:

Let $f(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$



Check that it's a density function: $f(x) \geq 0$, and

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_a^b \frac{1}{b-a}\,dx = \left.\frac{x}{b-a}\right|_{x=a}^{b} = \frac{b-a}{b-a} = 1 \quad \checkmark$$

Example 2:

Let $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} \lambda e^{-\lambda x}\,dx = \lim_{t\to\infty} \int_0^t \lambda e^{-\lambda x}\,dx = \lim_{t\to\infty} \left. -e^{-\lambda x}\right|_{x=0}^{t} = \lim_{t\to\infty} (e^{-\lambda t} + 1) = 1$$

More precisely, in example 1 we saw the density function for a uniform distribution.

Def:

A random variable X has the uniform distribution on an interval $[a,b]$ if X has the density function

$$f(x) = \begin{cases} \frac{1}{b-a} & , x \in [a,b] \\ 0 & , \text{otherwise} \end{cases}$$

We abreviate this as $X \sim \text{Unif}(a,b)$ or $\mathcal{U}(a,b)$

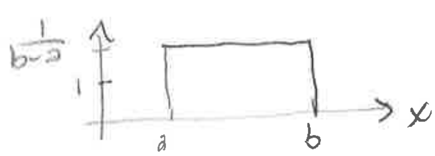
Similarly, if X has the density function

$$f(x) = \lambda e^{-\lambda x} \quad , x \geq 0$$

we say X has the exponential distribution. One common example of an "exp. dist." is waiting times btwn events.

* Note:

The value of a density function is not a probability. For the uniform distribution, $f(x)$ can be greater than 1 if, e.g. $b-a < 1$:



5.2: Distribution functions

Definition:

A r.v. (continuous, discrete, or in between) has a distribution function defined by $F(x) = P(X \le x)$. If X has a density function $f(x)$, then

$$F(x) = P(-\infty < X \le x) = \int_{-\infty}^{x} f(t)\,dt$$

Note:

$F(x)$ is aka the cumulative distribution function, or cdf.

The Return of FTC:

The distribution function gives use a nice property: note that for $a<b$,

$$P(X \le b) = P(X \le a) + P(a < X \le b)$$

$$\Rightarrow \underbrace{P(a < X \le b)}_{=\int_a^b f(x)\,dx} = \underbrace{P(X \le b) - P(X \le a)}_{=F(b)-F(a)}$$

$$\Rightarrow \int_a^b f(x)\,dx = F(b) - F(a)$$

Properties:

1. Nondecreasing: if $x_1 < x_2$, then $F(x_1) \le F(x_2)$
2. Limits at $\pm\infty$: $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$.

Example:

Find $F(x)$ for the expontial distribution.

We have that the density function is $f(x) = \lambda e^{-\lambda x}$; $x \geq 0$.

Then

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} \int_{-\infty}^{x} 0\,dt, & x \leq 0 \\ \int_{0}^{x} \lambda e^{-\lambda t}\,dt, & x \geq 0 \end{cases}$$

see why equality

$$= \begin{cases} 0, & x \leq 0 \\ -e^{-\lambda t}\Big|_{t=0}^{x}, & x \geq 0 \end{cases}$$

$$= \begin{cases} 0, & x \leq 0 \\ -e^{-\lambda x} + 1, & x \geq 0 \end{cases}$$

Note: This says that $P(X \leq x) = 1 - e^{-\lambda x}$, so $P(X > x) = 1 - F(x) = e^{-\lambda x}$.

Memoryless Prop

The exponential distribution lacks memory (or has the memoryless property): suppose T has an exponential dist. w/ par. λ. Then

$$P(T > t+s \mid T > t) = \frac{P(\{T > t+s\} \cap \{T > t\})}{P(T > t)} = \frac{P(T > t+s)}{P(T > t)}$$

$$= \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s}$$

$$= P(T > s)$$

EX: wait time if arrive at time t or time s has same prob.

Extensions of density + dist. fn's (5.1+5.2)

Recall:

Given a discrete r.v. X and a function $r(x)$, the expected value of $r(X)$ is

$$E\,r(X) = \sum_x r(x)\,P(X=x)$$

Def:

The expected value of a function of a continuous r.v. $r(X)$ is

$$E\,r(X) = \int_{-\infty}^{\infty} r(x)\,f(x)\,dx$$

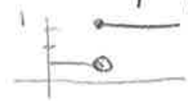
Note:

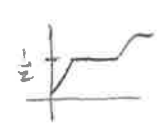
Moments and variance are defined as with discrete r.v.'s: for a cont. r.v. X,

$$r(x) = x^k \Rightarrow EX^k \text{ is the } k^{th} \text{ moment}$$

$$\mathrm{var}(X) = E(X-EX)^2 = EX^2 - (EX)^2$$

Now let's talk about median. Intuitively, the median is the value of the distribution $F(x)$ for which $F(x)$ crosses $\frac{1}{2}$. But $\{x : F(x) = \frac{1}{2}\}$ may be empty or contain ≥ 1 point





Definition:

A value m is a median for F is $P(X \leq m) \geq \frac{1}{2}$ and $P(X \geq m) \geq \frac{1}{2}$

Ex:

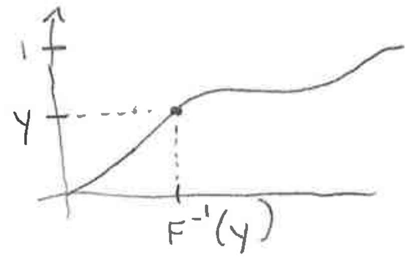
Inverse of cdf: the inverse of the cdf can is extremely useful if you're doing any statistical analysis of data.

Def:

The inverse distribution function (icdf), aka the quantile function, is the inverse of the distribution fuction $F(x)$:

$$F^{-1}(y) = \min\{x \mid F(x) \geq y\}$$

visually,



Why is this important? Because this gives us a way to draw randomly from any distribution, due to these pair of results:

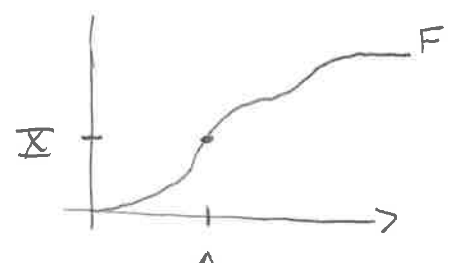
Theorem 5.2:
Suppose $f(x)$ has a continuous distribution. Then

$$Y = F(X) \sim Unif(0,1)$$

i.e., $P(F(X) < y) = P(X < F^{-1}(y)) = F(F^{-1}(y)) = y$

Theorem 5.3:
Suppose $U \sim Unif(0,1)$. Then $Y = F^{-1}(U)$ has the distribution function F.

So if you want to draw randomly from a general (even data) distr., simply draw an $X \sim Unif(0,1)$ and find the value of the distribution that matches it



$F^{-1}(X)$ is the random number sampled from the distribution F

Example:
$1 - e^{-\lambda x} = y \Rightarrow x = -\frac{\ln|1-y|}{\lambda} \Rightarrow U \sim Unif(0,1) \Rightarrow -\frac{\ln|1-U|}{\lambda}$ has the exp. dist.

Section 5.3: Functions of r.v.'s

Question:

What do we mean by distributions (and densities) of functions of r.v.'s?

EX:

Suppose $X \sim Exp(\lambda)$. What is dist. fn of $Y = X^2$?

Recall $P(X \leq x) = 1 - e^{-\lambda x}$.

$$\Rightarrow P(Y \leq y) = P(X^2 \leq y) = P(X \leq \sqrt{y}) = 1 - e^{-\lambda\sqrt{y}} \quad , y \geq 0$$

The density function is then the derivative wrt y:

$$f_Y(y) = \frac{d}{dy} P(Y \leq y) = \frac{\lambda}{2\sqrt{y}} e^{-\lambda\sqrt{y}} \quad , y \geq 0$$

In general, we can find the density of a fn. of a r.v. by differentiating.

Theorem:

Suppose X has the density f and $P(a < X < b) = 1$. Let $Y = r(X)$, where $r: (a,b) \to (\alpha, \beta)$ is cont. + strictly inc. Let $s: (\alpha,\beta) \to (a,b)$ be the inverse of r. Then Y has density

$$g(y) = f(s(y))\, s'(y) \quad , \quad y \in (\alpha, \beta)$$

* Do example using this explicitly

Section 5.4: Joint Distributions

For functions of two random variables.

Def:
Two r.v.'s X and Y are said to have joint density function $f(x,y)$ if $\forall A \subseteq \mathbb{R}^2$,

$$P((X,Y) \in A) = \iint_A f(x,y)\,dx\,dy,$$

where $f(x,y) \geq 0$ and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

Double integrals (aka iterated integrals)

Suppose $f(x,y) = x^2 + y^2$

$$\iint f(x,y)\,dx\,dy = \int\left(\int x^2+y^2\,dx\right)dy = \int \tfrac{1}{3}x^3 + y^2 x\,dy$$

$$= \tfrac{1}{3}x^3 y + \tfrac{1}{3}y^3 x + c$$

$f(x,y) = xy$

$$\iint xy\,dx\,dy = \int\left(\int xy\,dx\right)dy = \int y \int x\,dx\,dy$$

$$= \int y\left(\tfrac{1}{2}x^2\right)dy = \tfrac{1}{2}y^2\left(\tfrac{1}{2}x^2\right) + c = \tfrac{1}{4}x^2y^2 + c$$

<u>Example</u>:

Consider $f(x,y) = e^{-y}$, $0 < x < y < \infty$

f is a density fn:

$f(x,y) \geq 0 \quad \forall x,y$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^{\infty}\int_0^{y} e^{-y}\,dx\,dy$$

$$\boxed{\begin{aligned}\int_0^y e^{-y}\,dx &= xe^{-y}\Big|_{x=0}^{y}\\ &= ye^{-y} - 0\end{aligned}}$$

$$= \int_0^{\infty} ye^{-y}\,dy$$

$$= \lim_{t\to\infty} \int_0^t ye^{-y}\,dy$$

$u = y \qquad v = -e^{-y}$

$du = dy \qquad dv = e^{-y}dy$

$$= \lim_{t\to\infty} \left[-ye^{-y}\Big|_{y=0}^{t} + \int_0^t e^{-y}\,dy\right]$$

$$= \lim_{t\to\infty} \left[-\left(\frac{t}{e^t} - 0\right) - e^{-y}\Big|_{y=0}^{t}\right]$$

$$= -\lim_{t\to\infty} \frac{t}{e^t} - \lim_{t\to\infty}\left(\frac{1}{e^t} - e^0\right)$$

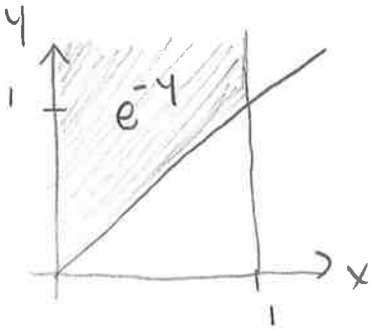
$$= -\lim_{t\to\infty} \frac{1}{e^t} - (0-1)$$

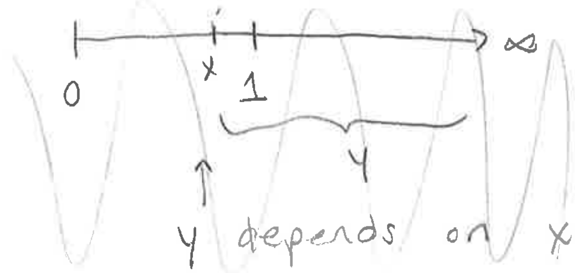
$$= 1$$

3/28/16

Now suppose we WTF $P(X \le 1) = P((X,Y) \in A)$, $A = \{(x,y) \mid x \le 1\}$.

$$P(X \le x) = \int^{1} \int e^{-y}\, d \; d$$





$(0 < x < y < \infty)$

$$= \int_0^1 \int_x^\infty e^{-y}\, dy\, dx$$

$$\boxed{\begin{aligned} \int_x^\infty e^{-y}\, dy &= \lim_{t \to \infty} \int_x^t e^{-y}\, dy \\ &= \lim_{t \to \infty} -e^{-y} \Big|_{y=x}^{t} \\ &= \lim_{t \to \infty} -\left(\frac{1}{e^t} - \frac{1}{e^x}\right) \\ &= \frac{1}{e^x} \end{aligned}}$$

$$= \int_0^1 e^{-y}\, dx = -e^{-x}\Big|_{x=0}^{1} = -(e^{-1} - e^{0}) = -\frac{1}{e} + 1$$

Now, given a distribution function, how do we find the joint density?

density? / distribution?

we describe joint distribution functions using

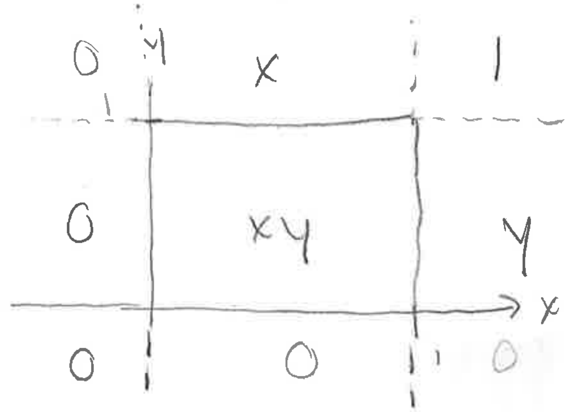
$$F(x,y) = P(X \le x, Y \le y)$$

Ex.

Suppose (X, Y) is unif. dist. over the square $\{(x,y) \mid 0<x<1,\ 0<y<1\}$. i.e.,

$$f(x,y) = 1 \quad , \quad 0<x,y<1$$

The dist. function needs to be found in pieces,



For $0 \le x \le 1$ and $0 \le y \le 1$:

$$P(X \le x, Y \le y) = \int_0^x \int_0^y 1 \, dv\, du$$

$$= xy$$

$$= F(x,y)$$

For $0 \le x \le 1$ and $y \ge 1$

$$P(X \le x, Y \le y) = P(X \le x, Y \le 1) = F(x,1) = x$$

For $x \ge 1$, $0 \le y \le 1$

$$P(X \le x, Y \le y) = F(1,y) = y$$

For $x \ge 1$, $y \ge 1$

$$F(1,1) = 1$$

$$\Rightarrow F(x,y) = \begin{cases} 0 & , \ x<0 \text{ or } y<0 \\ xy & , \\ x & , \\ y & , \\ 1 & , \end{cases}$$

3/28/18

Outside of the unit square?

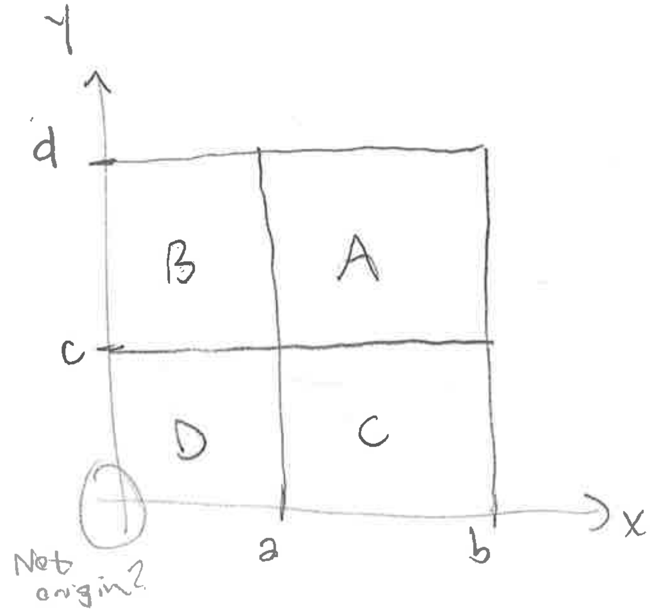
$P(a < X \le b,\ c < Y \le d)$

$= F(b,d)$

$- F(a,d)$

$- F(b,c)$

$+ F(a,c)$



$F(b,d) = A + B + C + D$

$-F(a,d) = -B \quad -D$

$-F(b,c) = \quad -C - D$

$+F(a,c) = \quad D$

The joint density fn.

$$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)\,dv\,du$$

$$\frac{\partial^2 F}{\partial x \partial y} = f$$

Section 5.5: Marginal and conditional distributions

<u>Recall</u>: for discrete r.v.'s, the marginal distributions can be obtained from a joint distribution by summing

$$P(X=x) = \sum_y P(\{X=x\} \cap \{Y=y\}) = \sum_y P(X=x, Y=y)$$

$$P(Y=y) = \sum_x P(X=x, Y=y)$$

Note. marginal cdf is $F_X = F_{XY}(x,\infty) = \lim_{y\to\infty} F_{XY}(x,y)$

<u>Definition</u>:
If the continuous r.v.'s X and Y have joint density $f(x,y)$, then the <u>marginal</u> <u>densities</u> of X and Y are given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \quad \text{and} \quad f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

<u>Note</u>:
Similarly to finding $P(X=x)$, the marginal distribution for a discrete r.v. (actually, this is more appropriately defined as the probability mass function), we 'sum' over the one of the variables and obtain a function for the other variable

<u>Example</u>:
Let X and Y have joint density function $f(x,y) = x+y$, for $0<x<1$ and $0<y<1$. Then the marginal density of X is

$$f_X(x) = \int_0^1 (x+y)\,dy = \left(xy + \tfrac{1}{2}y^2\right)\Big|_{y=0}^{1} = x + \tfrac{1}{2}, \quad 0 \le x \le 1$$

and Y is

$$f_Y(y) = \int_0^1 (x+y)\,dx = y + \tfrac{1}{2}, \quad 0 \le y \le 1$$

Question:
Are X and Y independent?

Def:
Two continuous r.v.'s X and Y are independent if

$$\underbrace{f(x,y)}_{\text{joint density}} = \underbrace{f_X(x)\, f_Y(y)}_{\text{marginal densities}}$$

Back to example:

$$f(x,y) = x+y$$

$$f_X f_Y = (x+\tfrac{1}{2})(y+\tfrac{1}{2}) = xy + \tfrac{1}{2}(x+y) + \tfrac{1}{4} \neq x+y \quad , \quad x,y \in [0,1]$$

$\rightarrow$ X and Y are dependent

Ex:

$$f(x,y) = \frac{y^{-3/2}\cos x}{e-1}\, e^{\sin x - \frac{2}{\sqrt{y}}} \quad , \quad 0 < x < \tfrac{\pi}{2}, \quad y > 0$$

We could integrate this, but there's a simpler approach:

Theorem:
If the joint density function $f(x,y)$ can be written as $g(x)h(y)$, then there is a constant c s.t.

$$f_X(x) = c\,g(x)$$

$$f_Y(y) = \frac{h(x)}{c}$$

Then $f(x,y) = f_X(x) f_Y(y)$, and is thus independent.

Back to example:

$$f(x,y) = \underbrace{\frac{\cos x\, e^{\sin x}}{e-1}}_{g(x)} \underbrace{\left(y^{-3/2} e^{-2/\sqrt{y}}\right)}_{h(y)}$$

Question:
where does the c come in? ← worksheet

Proof of Theorem:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int g(x)h(y)\,dy = g(x)\underbrace{\int h(y)\,dy}_{\equiv c}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = h(y)\underbrace{\int g(x)\,dx}_{\frac{1}{c}?}$$

$$1 = \iint f(x,y)\,dx\,dy = \int g(x)\,dx \int g(y)\,dy \implies \int g(x)\,dx = \frac{1}{c} \checkmark$$

Now let's look at conditional distributions. Again, recall the discrete case:

$$P(X = x \mid Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$ ← joint distribution / ← marginal distribution

prob. mass/density function

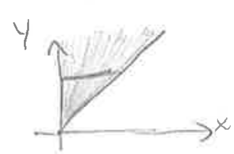
retcon: think Force> midichlorians

For the continuous case, we need the concept of a conditional density.

Def:
The conditional density of X, given $Y = y$, is

$$f_X(x \mid Y = y) = \frac{f(x,y)}{f_Y(y)} = \frac{f(x,y)}{\int f(t,y)\,dt}$$ ← joint density / ← marginal density

Example:



$f(x,y) = e^{-y} \quad 0 \le x \le y < \infty$

$$f_X(x \mid Y = y) = \frac{e^{-y}}{f_Y(y)}, \quad f_Y(y) = \int_0^y e^{-y}\,dx = xe^{-y}\Big|_{x=0}^{y} = ye^{-y}$$

$$= \frac{e^{-y}}{ye^{-y}} = \frac{1}{y}, \quad 0 < x < y$$

And

$$f_{\bar{Y}}(y \mid \bar{X}=x) = \frac{f(x,y)}{f_{\bar{X}}(x)} \quad , \quad f_{\bar{X}}(x) = \int_x^\infty e^{-y}\,dy = \lim_{s\to\infty} \int_x^s e^{-y}\,dy$$

$$= \frac{e^{-y}}{e^{-x}} \qquad\qquad = \lim_{s\to\infty} -e^{-y}\Big|_{y=x}^{s}$$

$$= e^{-(y-x)} \quad , \; y \geq x \qquad\qquad = \lim_{s\to\infty} \Big(\underbrace{-\tfrac{1}{e^s}}_{\to 0} + e^{-x} \Big)$$

Note:

We can use the definition of conditional and marginal densities to obtain the joint density:

$$f(x,y) = f_{\bar{X}}(x) f_{\bar{Y}}(y \mid \bar{X}=x) = f_{\bar{Y}}(y) f_{\bar{X}}(x \mid \bar{Y}=y)$$

Question:

For $f(x,y) = e^{-y}$, are $\bar{X}$ and $\bar{Y}$ independent?

No:

$$e^{-y} \neq y e^{-y} e^{-x}$$

In general, if the $\overbrace{\text{region where } f(x,y) > 0}^{\text{aka the support}}$ is not a rectangle, then $\bar{X}$ + $\bar{Y}$ dependent. Though we've seen an example where it is a rectangle and they were still dependent