Chapter 7: Option Pricing

In this chapter, we will discuss an investment topic, which will lead us into the last probability topic: Brownian motion.

First things first: what is an option?
Options are financial instruments, like stocks, that you can buy and sell. Let's talk specifically about stock options. A stock option is a contract that gives buyers the right (but not the obligation) to buy or sell the stock at an agreed-upon price during a certain period of time.

why do this?
- Options are frequently cheaper than stocks

  e.g. Facebook stock is $207.16 / stock

  option is ~$2-20 / stock

  people buy and sell options

- They can be less risky
- Higher percentage return potential

To better understand the relationship between options and stocks, let's consider a simple scenario

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Consider a $100 stock at time $n=0$ weeks. Then suppose it changes price according to

$n=0$	$n=1$	$n=2$	payoff
		$140	$40
	$120		
		$115	$15
$100			
		$120	$20
	$90		
		$80	$0

$$\text{payoff} = \max(X_2 - 100, 0)$$

A European call option with 'strike price' $100 and expiry 2 would be a contract giving you the option to buy (call $\leftrightarrow$ buy) a stock at $n=2$ for $100.

<u>Question:</u>
What should the option cost at time $n=0$?

Consider the payoffs for a portion of the tree. Option pricers want to eliminate the possibility of getting profit with zero probability of loss. Depending on the option price, this can happen!

$X_1 = 90$: $X_2 = 120 \leftrightarrow$ 'up' $\Rightarrow$ buy @$100 $\Rightarrow$ profit $= 120 - 100 - c$
$= 20 - c$

$X_2 = 80 \leftrightarrow$ 'down' $\Rightarrow$ don't buy @$100

$\Rightarrow$ profit $= -c$

But suppose we can buy stock or sell stock to balance our position: at $n=1$, the profits would be

	stock(x)	option(y)
If $X_2=120$	30	20-c
If $X_2=80$	-10	-c

$\Rightarrow 30x+(20-c)y = -10x+(-c)y$ ← profit is same regardless of stock action (up or down)

$\Rightarrow 40x+20y=0$

$\Rightarrow y=-2x$ (>0, buy, <0, sell)

If bought x, sell $2x=y$

If sold x, buy $2x=y$

$\Rightarrow$ profit is $-10x+2cx = (2c-10)x$

$c>5 \rightarrow$ large profit if buy x stock, sell 2x options

$c<5 \rightarrow$ large profit if sell x stock, buy 2x options

Def:
An arbitrage opportunity is a strategy that makes money without any possibility of a loss.

Note
So the price of this call option with absence of arbitrage is $c=5$

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Theorem:

Exactly one of the following holds:

i) $\exists$ betting scheme $\vec{x} = (x_1, \dots, x_n)$ s.t. $\sum_{i=1}^{m} x_i a_{i,j} \geq 0 \quad \forall j$

and $\sum_{i=1}^{m} x_i a_{i,k} > 0$ for some k

↑ arbitrage opp

ii) $\exists$ prob vector $\vec{p} = (p_1, \dots, p_n)$, $p_j > 0$, s.t.

$\sum_{j=1}^{n} a_{i,j} p_j = 0 \quad \forall i$

$\rightarrow$ $\vec{p}$ is called a martingale measure

If no arbitrage, $\exists$ p_j st.

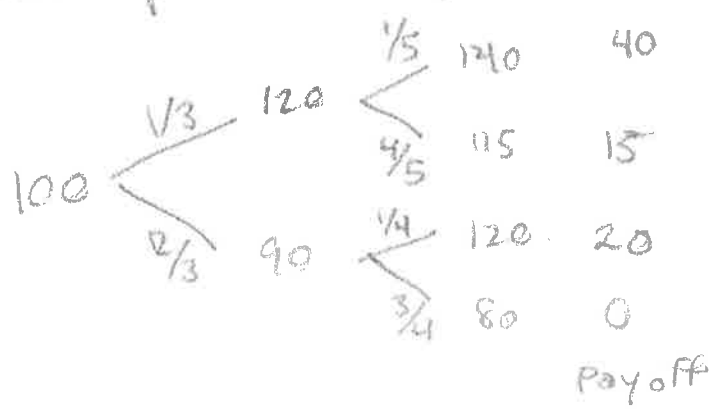
$30p_1 - 10p_2 = 0$ ① $\quad (20-c)p_1 + (-c)p_2 = 0$ ②

① $\rightarrow$ $30p_1 = 10p_2$

$$\left.\begin{aligned} p_1 &= \frac{p_2}{3} \\ p_1 + p_2 &= 1 \end{aligned}\right\} \rightarrow p_1 = \frac{1}{4},\ p_2 = \frac{3}{4}$$

② $\rightarrow$ $c = 20p_1 = 20\left(\frac{1}{4}\right) = 5$

Doing this procedure w/ the whole chart, we get



$\rightarrow c = \frac{1}{3}\left(\frac{1}{5}\right)(40) + \frac{1}{3}\left(\frac{4}{5}\right)(15) + \frac{2}{3}\left(\frac{1}{4}\right)(20) = 10$

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this pricing procedure is (I think) called martingale pricing – it's heavily used in quantitative finance, esp. for contracts w/ many stocks; solved using Monte Carlo

Section 7.2: Continuous Time

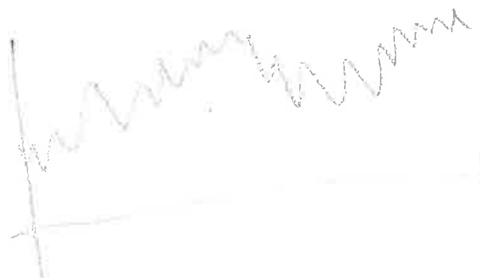
Brownian motion:

Let $X_1, X_2, \ldots$ be indep., s.t. $X_i = \begin{cases} -1, & p=\frac{1}{2} \\ 1, & p=\frac{1}{2} \end{cases}$

$\to E X_i = 0, \quad E X_i^2 = 1, \quad S_n = X_1 + \cdots + X_n$

$\frac{S_n}{\sqrt{n}} \to X \sim N(0,1)$

Let $t \geq 0$, consider $\frac{S_{[nt]}}{\sqrt{n}}$, $[nt] = \text{floor}(nt)$



Where are we going with this?

The Black-Scholes Fmla, for the price of a Euro. call option $\max(X_T - K, 0)$, is given by

$$E[\max(X_T - K, 0)]\big|_{t=0} = \int_K^\infty (X_T - K) P(X_T, T)\, dS$$

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where

$$\frac{\partial P}{\partial t} = \frac{1}{2}\sigma^2 X_T^2 \frac{\partial^2 P}{\partial X_T^2} + r X_T \frac{\partial P}{\partial X_T} - rP$$

r - risk-free interest rate,

~~K-strike~~

K - strike price

T - strike time