Adams Multistep Methods

The Adams family of multistep methods takes the solution to the differential equation $y'(t) = f(t, y)$,

$$y(t_{i+1}) = y(t_i) + \int_{t_i}^{t_{i+1}} f(t, y)\, dt,$$

and approximates $f(t, y)$ at certain points by an interpolating polynomial of degree q that matches the value of $f(t_k, y_k)$ for certain values of k. If we match

$$\begin{array}{lcc} k = i,\, i-1,\, i-2,\, \ldots,\, 1-q & \rightarrow & \text{Explicit Method (Adams-Bashforth)} \\ k = i+1,\, i,\, i-1,\, i-2,\, \ldots,\, 1-q+1 & \rightarrow & \text{Implicit Method (Adams-Moulton)} \end{array}$$

Notice that using an interpolating polynomial of degree q means we need to match the value of f at $s = q + 1$ points. Then we say that it is an $s-$step method. The general formula for creating an interpolating polynomial that matches the value of $f(t_k, y_k)$ forms what we call the Lagrange interpolating polynomial. For example, an interpolating polynomial at the points $f(t_{i-1}, y_{i-1})$, $f(t_i, y_i)$, and $f(t_{i+1}, y_{i+1})$, would be

$$f(t, y) \approx p(t) = \sum_k L_k(t) f_k = L_{i-1}(t) f(t_{i-1}) + L_i(t) f(t_i) + L_{i+1}(t) f(t_{i+1}),$$

where each $L_k(t)$ is a Lagrange interpolating polynomial.

Adams-Bashforth Methods

For explicit multistep methods, we can write

$$p(t) = \sum_{m=0}^{q} L_{i-m}(t) f(t_{i-m},\, y_{i-m}),$$

where

$$L_{i-m}(t) = \prod_{\substack{\ell=0 \\ \ell \neq m}}^{q} \frac{t - t_{i-\ell}}{t_{i-m} - t_{i-\ell}}.$$

Then the general iteration formula for an Adams-Bashforth method is

$$y_{i+1} = y_i + \int_{t_i}^{t_{i+1}} f(t, y)\, dt = y_i + \sum_{m=0}^{q} f(t_{i-m}, y_{i-m}) \int_{t_i}^{t_{i+1}} L_{i-m}(t)\, dt.$$

We discussed more details in class, such as how to initialize the method for all points t_{i-m} where $m > 0$, as well as how to derive AB2, so I'll point you toward the class notes for further discussion on Adams-Bashforth.

Adams-Moulton Methods

For implicit multistep methods, we need to incorporate the point (t_{i+1}, y_{i+1}) into our interpolating polynomial. If we want to sum over the same values of m, that means we need to change our indices:

$$p(t) = \sum_{m=0}^{q} L_{i+1-m}(t) f(t_{i+1-m},\, y_{i+1-m}),$$

where

$$L_{i+1-m}(t) = \prod_{\substack{\ell=0 \\ \ell \neq m}}^{q} \frac{t - t_{i+1-\ell}}{t_{i+1-m} - t_{i+1-\ell}}. \tag{1}$$

Then the general iteration formula for an Adams-Moulton method is

$$y_{i+1} = y_i + \int_{t_i}^{t_{i+1}} f(t,y)\,dt = y_i + \sum_{m=0}^{q} f(t_{i+1-m}, y_{i+1-m}) \int_{t_i}^{t_{i+1}} L_{i+1-m}(t)\,dt.$$

As an example, suppose we want to calculate the Adams-Moulton method for an interpolating polynomial of degree $q = 3$, i.e.

$$f(t,y) \approx p(t) = L_{i-2}(t) f(t_{i-2}, y_{i-2}) + L_{i-1}(t) f(t_{i-1}, y_{i-1}) + L_i(t) f(t_i, y_i) + L_{i+1}(t) f(t_{i+1}, y_{i+1}).$$

The iteration formula for this $(q+1 = 4)-$step Adams-Moulton method, aka AM4, is given by

$$\begin{aligned}
y_{i+1} = y_i &+ \sum_{m=0}^{q} f(t_{i+1-m}, y_{i+1-m}) \int_{t_i}^{t_{i+1}} L_{i+1-m}(t)\,dt \\
= y_i &+ f(t_{i+1}, y_{i+1}) \int_{t_i}^{t_{i+1}} L_{i+1}(t)\,dt + f(t_i, y_i) \int_{t_i}^{t_{i+1}} L_i(t)\,dt \\
&+ f(t_{i-1}, y_{i-1}) \int_{t_i}^{t_{i+1}} L_{i-1}(t)\,dt + f(t_{i-2}, y_{i-2}) \int_{t_i}^{t_{i+1}} L_{i-2}(t)\,dt
\end{aligned}$$

Here, the Lagrange interpolating polynomials can be found by substituting the appropriate value of m into the product for $L_{i+1-m}(t)$ in (1):

$$\begin{aligned}
L_{i+1}(t) &= \frac{(t - t_{i-2})(t - t_{i-1})(t - t_i)}{(t_{i+1} - t_{i-2})(t_{i+1} - t_{i-1})(t_{i+1} - t_i)} = \frac{(t - t_{i-2})(t - t_{i-1})(t - t_i)}{(3h)(2h)(h)} \\
L_i(t) &= \frac{(t - t_{i-2})(t - t_{i-1})(t - t_{i+1})}{(t_i - t_{i-2})(t_i - t_{i-1})(t_i - t_{i+1})} = \frac{(t - t_{i-2})(t - t_{i-1})(t - t_{i+1})}{(2h)(h)(-h)} \\
L_{i-1}(t) &= \frac{(t - t_{i-2})(t - t_i)(t - t_{i+1})}{(t_{i-1} - t_{i-2})(t_{i-1} - t_i)(t_{i-1} - t_{i+1})} = \frac{(t - t_{i-2})(t - t_i)(t - t_{i+1})}{(h)(-h)(-2h)} \\
L_{i-2}(t) &= \frac{(t - t_{i-1})(t - t_i)(t - t_{i+1})}{(t_{i-2} - t_{i-1})(t_{i-2} - t_i)(t_{i-2} - t_{i+1})} = \frac{(t - t_{i-1})(t - t_i)(t - t_{i+1})}{(-h)(-2h)(-3h)}
\end{aligned}$$

Seeing them written out helps us point out a pattern for the Lagrange polynomials, that in the numerator we multiply all the products of $(t - t_k)$ for the points we're interpolating at *except* for the point of interest (e.g., $k = i+1$ in $L_{i+1}(t)$). Keep this in mind when substituting in the product (1), to help keep the terms straight!

To integrate these polynomials, it simplifies the evaluation of the integral to make an appropriate substitution to transform the limits of integration from t_i to t_{i+1}, to 0 to 1: the substitution $u = \dfrac{t - t_i}{h}$ accomplishes this transformation. After integrating, the integrals $\int L_{i+1-m}(t)\,dt$ give us appropriate coefficients for our iteration formula. For AM4 ($q = 3$), we obtain

$$y_{i+1} = y_i + \frac{h}{24}\left(9f(t_{i+1}, y_{i+1}) + 19f(t_i, y_i) - 5f(t_{i-1}, y_{i-1}) + f(t_{i-2}, y_{i-2})\right)$$