Part II: Dynamical Systems

In optimization, we used methods to find the max or min of an objective function, with no time variation but possibly with constraints.

In dynamical systems, we use methods to find how the behavior of a time-varying system of equations changes over time.

Example: Whale problem

$$\dot{x} = r_1 x\left(1-\frac{x}{K_1}\right) - \alpha_1 xy := f_1(x,y) \quad \leftarrow \text{blue}$$

$$\dot{y} = r_2 y\left(1-\frac{y}{K_2}\right) - \alpha_2 xy := f_2(x,y) \quad \leftarrow \text{fin}$$

where
- $r_1 = 0.05$ and $r_2 = 0.08$ are intrinsic growth rates
- K_1, K_2 are the carrying capacities of each pop. in the absence of competition
- α_i are the competition rate

We're interested in the steady states of this system - where the population sizes go to when they're allowed to change over time.

Case 1: $\alpha_1 = \alpha_2 = 0$ (No competition)

$$\dot{x} = r_1 x\left(1-\frac{x}{K_1}\right)$$

$$\dot{y} = r_2 y\left(1-\frac{y}{K_2}\right)$$

Each equation becomes the logistic equation.

Sometimes differential equations can be solved analytically:

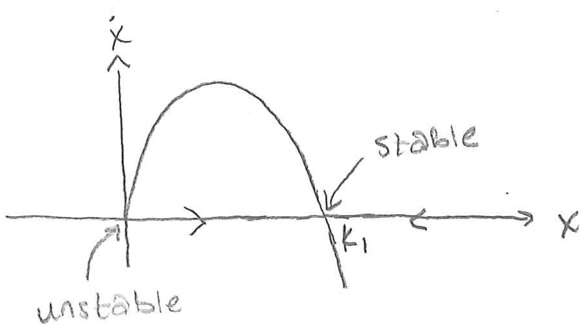
$$\int \frac{1}{x\left(1-\frac{x}{K_1}\right)}dx = \int r_1 dt \quad \rightarrow \quad x = \frac{K_1}{1+\left(\frac{K_1 - x_0}{x_0}\right)e^{-r_1 t}} \qquad (*)$$

where $x_0 = x(0)$.

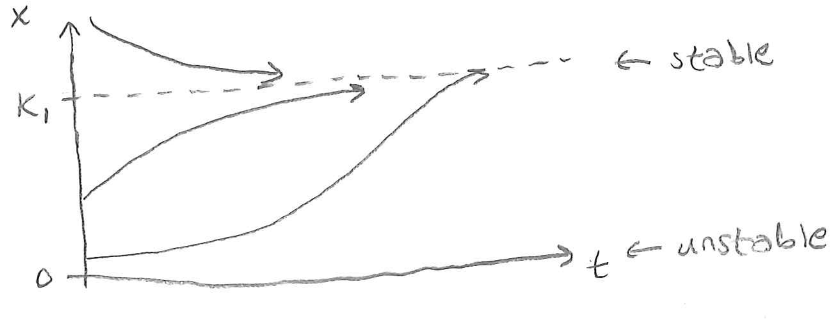
Alternatively, (and when we can't solve a DE analytically), we can perform a steady-state analysis. Let $\dot{x}=0$ and $\dot{y}=0$. Then

$$\begin{cases} 0 = r_1 x\left(1-\frac{x}{K_1}\right) \\ 0 = r_2 y\left(1-\frac{y}{K_2}\right) \end{cases} \Rightarrow \begin{matrix} x=0 \text{ or } x=K_1 \\ y=0 \text{ or } y=K_2 \end{matrix}$$

Stability: look at the slope of $\dot{x}$ in relation to some starting point x (the phase plane):



Let's check with our analytical solution $(*)$:



Now let's look at reintroducing competition. This 'couples' the two differential equations and makes them nonlinear.

Definition:

A (continuous time) dynamical system consists of n state variables $(x_1, \dots, x_n)$ and a system of n differential equations

$$\frac{dx_1}{dt} = f_1(x_1, \dots, x_n)$$

$$\vdots$$

$$\frac{dx_n}{dt} = f_n(x_1, \dots, x_n)$$

defined on the state space $(x_1, \dots, x_n) \in S \subseteq \mathbb{R}^n$

Definition:

A state $(x_1^*, \dots, x_n^*)$ is called an equilibrium point if

$$f_1(x_1^*, \dots, x_n^*) = 0$$

$$\vdots$$

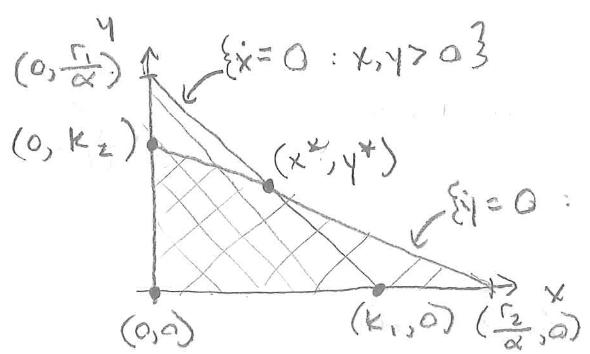
$$f_n(x_1^*, \dots, x_n^*) = 0$$

Case 2: $\alpha := \alpha_1 = \alpha_2 \neq 0$

Let's look at the steady states:

$$f_1(x,y) = 0 = x\left(r_1\left(1 - \frac{x}{K_1}\right) - \alpha y\right) \quad \Rightarrow \quad x = 0 \text{ or } x = x^*$$

$$f_2(x,y) = 0 = y\left(r_2\left(1 - \frac{y}{K_2}\right) - \alpha x\right) \qquad y = 0 \text{ or } y = y^*$$

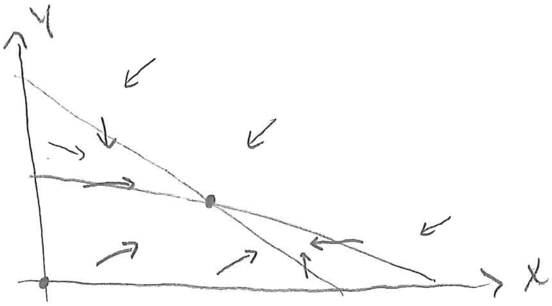


/// : $\dot{y} > 0$

\\\ : $\dot{x} > 0$

'phase space' or 'state space'

We can look at the relative signs of $\dot{x}, \dot{y}$ to see where solutions go

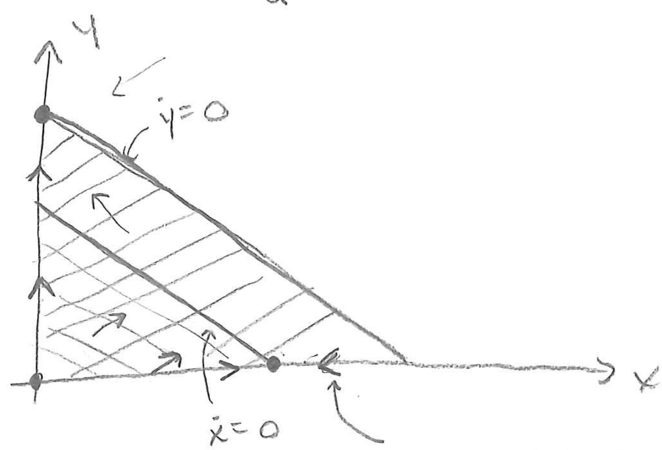


How might this change as we vary parameters?

Is there any value of α for which (x^*, y^*) does not exist?

α increasing $\rightarrow$ the points $(0, \frac{r_1}{\alpha})$, $(\frac{r_2}{\alpha}, 0)$ move

Let $\frac{r_1}{\alpha} < k_2$ and $\frac{r_2}{\alpha} < k_1$. Then



$y=0, x \neq 0 \Rightarrow \dot{x} = x r_1 (1 - \frac{x}{k_1})$
$\dot{y} = 0$

Note:

We call an equilibrium $(x_1^*, \ldots, x_n^*) \in \mathbb{R}^n$ a state (and why we call $x_1, \ldots, x_n$ state variables because the state of the system is enough to determine its future behavior, barring external forces acting on the system.

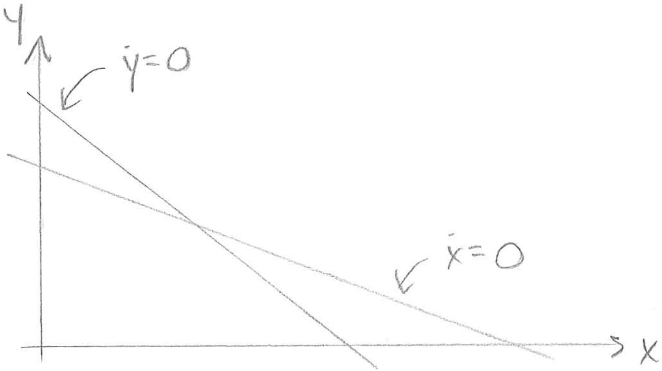
↳ Picard's existence theorem applies.

Last time: whale problem

$$\dot{x} = r_1 x\left(1 - \frac{x}{K_1}\right) - \alpha x y$$

$$\dot{y} = r_2 y\left(1 - \frac{y}{K_2}\right) - \alpha x y$$

State space:



The lines/curves where $\dot{x}=0$ or $\dot{y}=0$ are called nullclines, or isoclines. Inspecting nullclines is a geometric method of finding equilibria and their stability. Let's investigate an analytical method of determining stability.

Stability Analysis

In 1D, we looked at the sign of the RHS of a differential equation. In general, we only need to look at the slope of the RHS at the equilibrium point

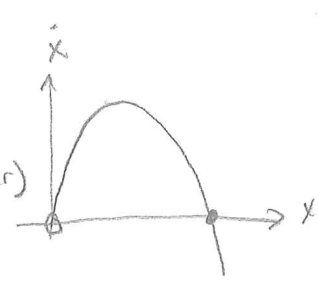
Theorem:

Let $\dot{x} = f(x)$, where $f(x)$ is differentiable. Then, for an equilibrium point x^* such that $f(x^*) = 0$, we say x^* is

i) stable if $f'(x^*) < 0$

ii) unstable if $f'(x^*) > 0$

EX: (from earlier)



In higher dimensions, we look at the eigenvalues and eigenvectors of the system.

Example: Predator-prey (foxes-rabbits)
Rabbits eat grass and foxes eat rabbits.
what is a reasonable model to make?

$$\left.\begin{aligned} \text{rabbits: } & \dot{x} = ax \\ \text{foxes: } & \dot{y} = -cy \end{aligned}\right\} (*)$$

← without predation rabbits' pop. grows without bound ($x(t) = e^{at} + c$)

← Without prey, foxes die off

Now, let's couple the equations:

$$(**) \begin{cases} \dot{x} = ax - \gamma xy \\ \dot{y} = -cy + \delta xy \end{cases}$$

← Rabbit pop. decreases by some amount proportional to both populations.

← Foxes increase proportionally to both populations.

Equlibria of $(**)$:

$$\begin{cases} 0 = x(a - \gamma y) \\ 0 = y(\delta x - c) \end{cases} \longrightarrow \begin{matrix} x = 0, & y = \frac{a}{\gamma} \\ y = 0, & x = \frac{c}{\delta} \end{matrix} \rightarrow (0,0), \left(\frac{c}{\delta}, \frac{a}{\gamma}\right)$$

Stability of $(*)$:

For the linear system we can do linear stability analysis at $(x^*, y^*) = (0,0)$: solve the linear system of ODE's by starting with an ansatz (an initial guess to the form of the solution).

Ansatz: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\sigma t}$ ← straight lines, traveled along at speed σ

$$\overset{(*)}{\longrightarrow} \sigma \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\sigma t} = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\sigma t}$$

$$\rightarrow 0 = \begin{pmatrix} a-\sigma & 0 \\ 0 & -c-\sigma \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\sigma t} \longrightarrow 0 = \begin{pmatrix} a-\sigma & 0 \\ 0 & -c-\sigma \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

So $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\sigma t}$ is a solution if $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ is an eigenvector of the matrix $\begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}$, with eigenvalue σ. Eigenvalues of a system $Ax = \lambda x$ are given by solutions λ to the characteristic equation, $\det(A - \lambda I) = 0$. Here,

$$\det \begin{pmatrix} a-\sigma & 0 \\ 0 & -c-\sigma \end{pmatrix} = 0 \rightarrow (a-\sigma)(-c-\sigma) = 0$$

$$\rightarrow \sigma = a, -c$$

with eigenvectors:

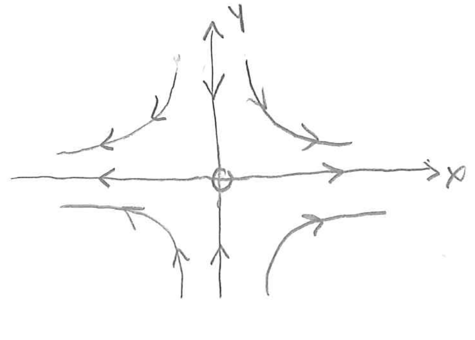
$\sigma = a$: $\begin{pmatrix} a-a & 0 \\ 0 & -c-a \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \vec{0} \rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\sigma = -c$: $\begin{pmatrix} a+c & 0 \\ 0 & -c+c \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \vec{0} \rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

So the solutions to $(*)$ are

$$\begin{pmatrix} x \\ y \end{pmatrix} = d_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{at} + d_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-ct} = \begin{pmatrix} d_1 e^{at} \\ d_2 e^{-ct} \end{pmatrix}$$

We are interested in the behavior of these solutions as $t \to \infty$:



$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ has $\sigma = a$

$\Rightarrow$ repelling in the x-direction

$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ has $\sigma = -c$

$\Rightarrow$ attracting in the y-direction we call this type of equilibrium a saddle point. It is unstable.

In general, we classify possible behaviors by looking at the possible eigenvalues we can get. For a linear system $\dot{\vec{x}} = A\vec{x}$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the characteristic equation is

$$\det(A - \lambda I) = 0$$

where λ is the unknown eigenvalue.

$$\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0 \iff \lambda^2 - \tau\lambda + \Delta = 0$$

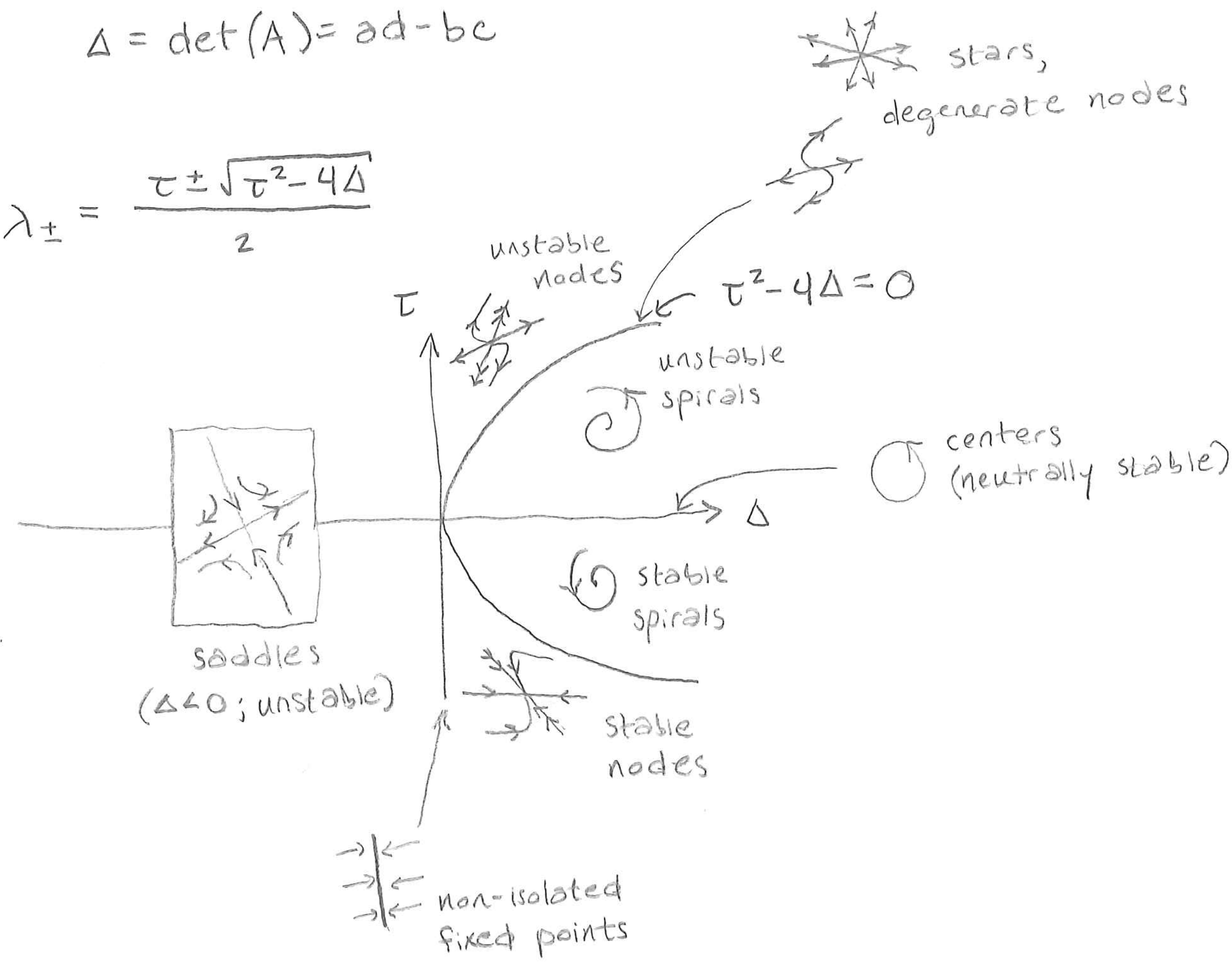
where τ is the trace of A and Δ is the determinant,

$$\tau = \text{trace}(A) = a + d$$

$$\Delta = \det(A) = ad - bc$$

Then

$$\lambda_\pm = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$



In general, we may have nonlinear systems. In that case, we first linearize the system about the equilibrium point, then do our linear stability analysis.

Consider

$$\dot{\vec{x}} = \vec{f}(\vec{x}) \quad , \quad \vec{x} \in \mathbb{R}^n \ , \ f: \mathbb{R}^n \to \mathbb{R}^n$$

$$\vec{f} = (f_1, f_2, \dots, f_n)$$

$$\vec{x} = (x_1, x_2, \dots, x_n)$$

Equilibrium points (aka steady states, or fixed points) are $\vec{x}^*$ such that $f(\vec{x}^*) = 0$. For linear stability, consider a small perturbation about $x = x^*$,

$$x = x^* + y \quad , \quad \|y\| \ll 1$$

$\rightarrow$ LHS: $\frac{dx}{dt} = \frac{d}{dt}(x^* + y) = \frac{dy}{dt}$

RHS: $f(x^* + y) \approx \cancel{f(x^*)}^{\,0} + Df(x^*)(x - x^*) + \underbrace{\cdots}_{\text{linear stability calculations throw these out}}$

$\uparrow$ Taylor expand about x^*

$$\rightarrow \frac{dy}{dt} = Df(x^*)(x - x^*)$$

$$= \underbrace{Df(x^*)}_{\text{Jacobian } (J)} y$$

So the linear stability problem is $\dot{y} = J(x^*) y$. Solutions to this look like

$$\vec{y} = t^k e^{\lambda t} \vec{v}$$

where $\lambda =$ an eigenvalue of $J(x^*)$

$\vec{v} =$ an eigenvector or generalized eigenvector of $J(x^*)$

Note: If $Re(\lambda)>0$ for any λ, then x^* is unstable

$Re(\lambda)<0$ $\forall \lambda$, x^* is stable

Example: Fox-rabbit problem

$$\dot{x} = ax - \gamma xy$$
$$\dot{y} = -cy + \delta xy$$

For $(x^*, y^*) = (0,0)$:

let $x = 0 + \tilde{x}$ $\rightarrow$ $\dot{\tilde{x}} = a(0+\tilde{x}) - \gamma(0+\tilde{x})(0+\tilde{y}) = a\tilde{x} + \cdots$

$y = 0 + \tilde{y}$ $\quad$ $\dot{\tilde{y}} = -c(0+\tilde{y}) + \delta(0+\tilde{x})(0+\tilde{y}) = -c\tilde{y} + \cdots$

For $(x^*, y^*) = (\frac{c}{\delta}, \frac{a}{\gamma})$:

let $x = \frac{c}{\delta} + \tilde{x}$ $\rightarrow$ $\dot{\tilde{x}} = a(\frac{c}{\delta} + \tilde{x}) - \gamma(\frac{c}{\delta} + \tilde{x})(\frac{a}{\gamma} + \tilde{y})$

$y = \frac{a}{\gamma} + \tilde{y}$ $\quad$ $\dot{\tilde{y}} = -c(\frac{a}{\gamma} + \tilde{y}) - \delta(\frac{c}{\delta} + \tilde{x})(\frac{a}{\gamma} + \tilde{y})$

$$\dot{\tilde{x}} = \cancel{\frac{ac}{\delta}} + \cancel{a\tilde{x}} - \gamma\left(\cancel{\frac{ac}{\delta\gamma}} + \frac{c}{\delta}\tilde{y} + \cancel{\frac{a}{\gamma}\tilde{x}} + \underbrace{\tilde{x}\tilde{y}}_{\text{disregard} \rightarrow \text{h.o.t.}}\right) = -\frac{c\gamma}{\delta}\tilde{y}$$

$$\dot{\tilde{y}} = \frac{a\delta}{\gamma}\tilde{x}$$

Notice:

$$\begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{y}} \end{pmatrix} = J\left(\frac{c}{\delta}, \frac{a}{\gamma}\right)\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$$

18/3

Stability:

$$\det(J-\lambda I)=0 \rightarrow \det\begin{bmatrix} -\lambda & -\frac{c\gamma}{\delta} \\ \frac{a\delta}{\gamma} & -\lambda \end{bmatrix}=0$$

$$\rightarrow \lambda^2 + ca = 0$$

$$\rightarrow \lambda = \pm i\sqrt{ca}$$

Classify equilibrium:

$$\tau(J)=0 \qquad \Rightarrow \text{center}$$
$$\Delta(J) = 0 + ca$$

How do solutions behave?

For the <u>linearized</u> system, solutions have the form

$$\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix} = \begin{pmatrix}c_1\\ c_2\end{pmatrix} e^{i\sqrt{ca}\,t} + \begin{pmatrix}c_3\\ c_4\end{pmatrix} e^{-i\sqrt{ca}\,t}$$

Find eigenvectors from

$$(J-\lambda I)\begin{pmatrix}c_1\\ c_2\end{pmatrix}=0$$

$$\rightarrow \lambda = +i\sqrt{ca}: \quad \begin{pmatrix}c_1\\ c_2\end{pmatrix} = \begin{pmatrix}\frac{i\gamma\sqrt{c}}{\delta}\\ \sqrt{a}\end{pmatrix} \quad \rightarrow \quad \begin{aligned}\tilde{x} &= \frac{i\gamma\sqrt{c}}{\delta}\left(e^{i\sqrt{ac}\,t} - e^{-i\sqrt{ac}\,t}\right)\\ \tilde{y} &= \sqrt{a}\left(e^{i\sqrt{ac}\,t} + e^{-i\sqrt{ac}\,t}\right)\end{aligned}$$

$$\lambda = -i\sqrt{ca}: \quad \begin{pmatrix}c_3\\ c_4\end{pmatrix} = \begin{pmatrix}-\frac{i\gamma\sqrt{c}}{\delta}\\ \sqrt{a}\end{pmatrix}$$

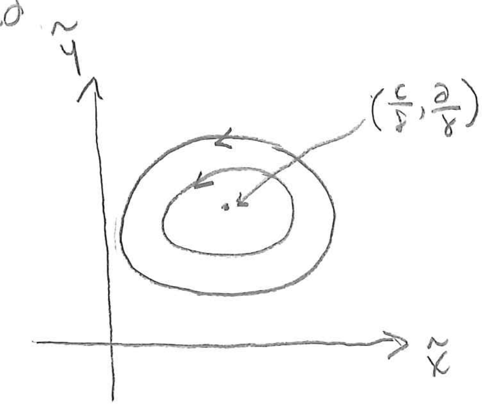
since $$\frac{e^{i\sqrt{ca}\,t}+e^{-i\sqrt{ca}\,t}}{2} = \cos(\sqrt{ca}\,t)$$

$$\frac{e^{i\sqrt{ca}\,t}-e^{-i\sqrt{ca}\,t}}{2i} = \sin(\sqrt{ca}\,t)$$

we can describe the solutions (to the linearized system) as

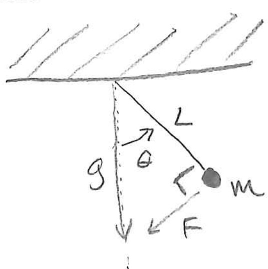
$$\tilde{x} = \frac{2\gamma\sqrt{c}}{\delta}\sin(\sqrt{ca}\,t)$$

$$\tilde{y} = 2\sqrt{a}\cos(\sqrt{ca}\,t)$$



Interpretation: this gives a continuous family of neutrally stable cycles. In fact this is the Lotka-Volterra predator-prey model, a classical model in mathematical biology. One drawback of the model is that cycles in predator-prey systems typically have one characteristic amplitude, not an infinite amount of possibilities.

Example: Pendulum



$$F = -m\underbrace{g}_{\text{accelleration due to gravity}}\sin\theta$$

$$= m\underbrace{\frac{d^2s}{dt^2}}_{\text{displacement, } s=L\theta \text{ is arc length}} = mL\ddot{\theta}$$

$\rightarrow \cancel{m}L\ddot{\theta} = -\cancel{m}g\sin\theta$

$\rightarrow \ddot{\theta} = -\frac{g}{L}\sin\theta$

Write as a first-order system: let $\Omega = \dot{\theta}$. Then

$\dot{\theta} = \Omega$

$\dot{\Omega} = -\frac{g}{L}\sin\theta$

Equilibria:

$$0 = \Omega \qquad \qquad \rightarrow (\theta, \Omega) = (k\pi, 0), \quad k \in \mathbb{Z}$$
$$0 = -\frac{g}{L}\sin\theta$$

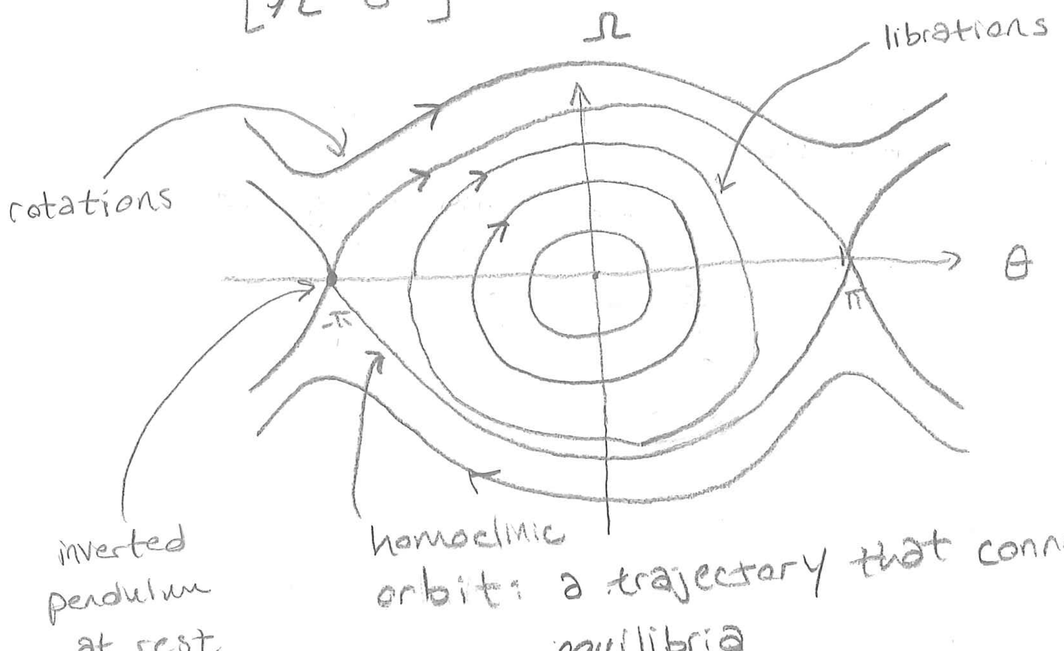
Look at $(\Omega, \theta) = (0,0)$:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial \theta} & \frac{\partial f_1}{\partial \Omega} \\ \frac{\partial f_2}{\partial \theta} & \frac{\partial f_2}{\partial \Omega} \end{bmatrix}\Bigg|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix}$$

$$\tau(J) = 0 \qquad \rightarrow (0,0) \text{ is a center}$$
$$\Delta(J) = \frac{g}{L}$$

Now consider $(\Omega, \theta) = (0, \pi)$:

$$J = \begin{bmatrix} 0 & 1 \\ g/L & 0 \end{bmatrix} \rightarrow (\pi, 0) \text{ is a saddle}$$



homoclinic orbit: a trajectory that connects two equilibria

<u>Summary</u>: Linear stability analysis for continuous-time systems

$$\dot{x} = f_1(x,y)$$
$$\dot{y} = f_2(x,y)$$

Equilibrium points: solve $f_1(x,y)=0$ and $f_2(x,y)=0$ for x and y

Stability: linearize about x^*, y^*

$$x = x^* + \tilde{x}$$
$$y = y^* + \tilde{y}$$

leads to the linearized system

$$\begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{y}} \end{pmatrix} = A \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \quad , \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = J(x^*, y^*)$$

whose solution is

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = k_1 \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\lambda_1 t} + k_2 \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} e^{\lambda_2 t}$$

where λ_i are the eigenvalues and $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ are the eigenvectors

Stability follows from considering the sign of the λ_i's

Characterize the equilibrium point using

$$\tau(A) = \lambda_1 + \lambda_2$$
$$\Delta(A) = \lambda_1 \lambda_2$$

Discrete Dynamical Systems

A discrete dynamical system is one in which the system behavior is dictated at discrete points in time. For example,

$$x(t_{n+1}) = x_{n+1} = f_1(x_n, y_n) \qquad \Delta x = x_{n+1} - x_n = g_1(x_n, y_n)$$

OR

$$y(t_{n+1}) = y_{n+1} = f_2(x_n, y_n) \qquad \Delta y = y_{n+1} - y_n = g_2(x_n, y_n)$$

We determine equilibria using the same methods. Let $\Delta x = 0$ and $\Delta y = 0$. Then x^*, y^* satisfy

$$0 = g_1(x^*, y^*)$$

$$0 = g_2(x^*, y^*)$$

Again, stability comes from analyzing the linearized system: let

$$x_{n+1} = x^* + \tilde{x}_{n+1} \quad , \quad |\tilde{x}_{n+1}|, |\tilde{y}_{n+1}| \ll 1$$

$$y_{n+1} = y^* + \tilde{y}_{n+1}$$

Then we can write the linearized system as

$$\begin{pmatrix} \tilde{x}_{n+1} \\ \tilde{y}_{n+1} \end{pmatrix} = A \begin{pmatrix} \tilde{x}_n \\ \tilde{y}_n \end{pmatrix}$$

If all eigenvalues λ of A satisfy $|\lambda| > 1 \rightarrow$ unstable

$|\lambda| < 1 \rightarrow$ stable

Example:

$$x_{n+1} = f(x_n) \rightarrow \underbrace{\tilde{x}_{n+1}}_{(\tilde{x}_{n+1} - x^*)} = f(x^* + \tilde{x}_n) \approx \cancel{f(x^*)}^{\,0} + f'(x^*)\tilde{x}_n + \cdots$$

If $|f'(x^*)| = |\lambda| > 1$, $\tilde{x}_{n+1} > \tilde{x}_n \rightarrow x^* + \tilde{x}_n$ gets bigger

< 1, $\tilde{x}_{n+1} < \tilde{x}_n \rightarrow x^* + \tilde{x}_n \rightarrow x^*$

Note: This linearization tells us nothing about when $|f'(x^*)| = 1$. If so, the neglected terms determine local stability

To get more intuition about stability in 2D, let's look at solutions to the linearized system. We start with an ansatz, i.e. a guess to the form of the solution:

$$\begin{pmatrix} \tilde{x}_n \\ \tilde{y}_n \end{pmatrix} = k_1 \underbrace{\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + k_2 \begin{pmatrix} c_3 \\ c_4 \end{pmatrix}}_{\text{eigenvectors}}, \quad k_1, k_2 \text{ unknown constants}$$

$$\rightarrow \begin{pmatrix} \tilde{x}_{n+1} \\ \tilde{y}_{n+1} \end{pmatrix} = k_1 A \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + k_2 A \begin{pmatrix} c_3 \\ c_4 \end{pmatrix} \underset{A\vec{x} = \lambda\vec{x}}{=} k_1 \lambda_1 \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + k_2 \lambda_2 \begin{pmatrix} c_3 \\ c_4 \end{pmatrix}$$

$$\vdots$$

$$\begin{pmatrix} \tilde{x}_{n+j} \\ \tilde{y}_{n+j} \end{pmatrix} = k_1 \lambda_1^j \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + k_2 \lambda_2^j \begin{pmatrix} c_3 \\ c_4 \end{pmatrix}$$

So if $|\lambda_1|, |\lambda_2| < 1$, then $\begin{pmatrix} \tilde{x}_{n+j} \\ \tilde{y}_{n+j} \end{pmatrix} \to 0$ as $j \to \infty$.

Note: we can compare this to the continuous time case.

$$\begin{pmatrix} \frac{d\tilde{x}}{dt} \\ \frac{d\tilde{y}}{dt} \end{pmatrix} = A \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \rightarrow \begin{pmatrix} \dfrac{\tilde{x}_{t+\Delta t} - \tilde{x}_t}{(t+\Delta t) - t} \\ \dfrac{\tilde{y}_{t+\Delta t} - \tilde{y}_t}{\Delta t} \end{pmatrix} = A \begin{pmatrix} \tilde{x}_t \\ \tilde{y}_t \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} \tilde{x}_{t+\Delta t} \\ \tilde{y}_{t+\Delta t} \end{pmatrix} = \begin{pmatrix} \tilde{x}_t \\ \tilde{y}_t \end{pmatrix} + A\,\Delta t \begin{pmatrix} \tilde{x}_t \\ \tilde{y}_t \end{pmatrix} = (I + A\Delta t)\begin{pmatrix} \tilde{x}_t \\ \tilde{y}_t \end{pmatrix}$$

Stability:

$$(I + A\Delta t)\vec{x} = \lambda_d \vec{x}$$

$$= (1 + \lambda_c \Delta t)\vec{x}$$

where λ_d is an eigenvalue of the discrete system

λ_c is an eigenvalue of the continuous system

Example: logistic equation

Recall the whole problem in 1D,

$$\frac{dx}{dt} = r_1 x\left(1 - \frac{x}{K_1}\right)$$

We had a stable equilibrium at $x^* = K_1$. Consider a similar discrete system,

$$x_{n+1} = \alpha x_n (1 - x_n), \quad \alpha > 1 \qquad \text{(the logistic map)}$$

Equilibria:

$$x_n = \alpha x_n (1 - x_n) \rightarrow x^* = 0 \quad \text{or} \quad 1 = \alpha(1 - x^*)$$

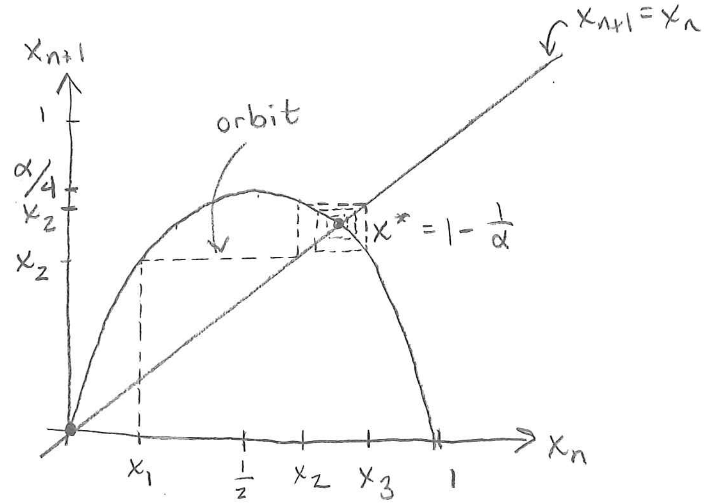
$$\frac{1}{\alpha} = 1 - x^*$$

$$x^* = 1 - \frac{1}{\alpha}$$

We can determine stability either by finding the eigenvalues, as above, or graphically using something called a cobweb plot.

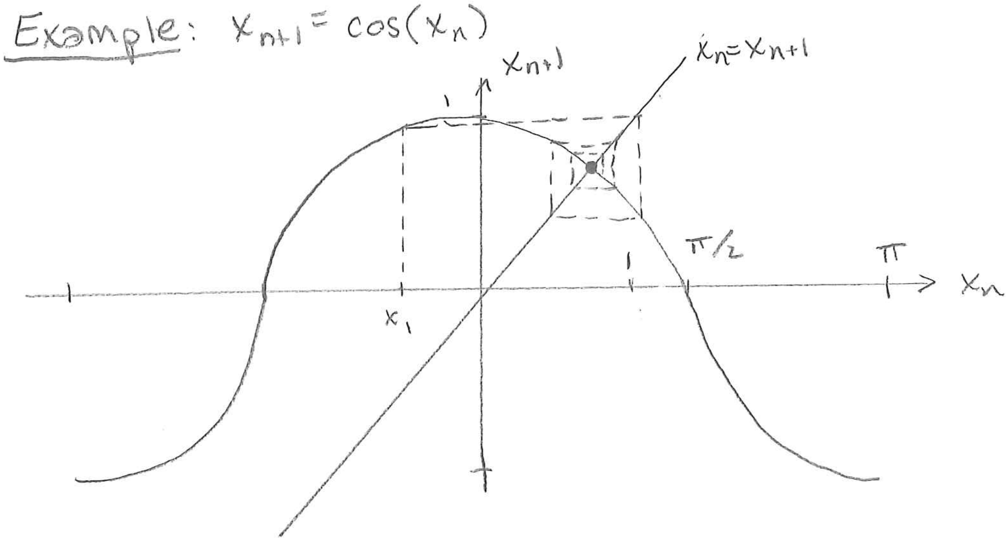
20/4

To construct a cobweb plot, remember that equilibria are points such that $x_{n+1} = x_n$:



So an initial condition x_n will 'map' to some x_{n+1}, which becomes the x_n of the next iteration, and this process repeats until x_{n+1} either approaches a (stable) equilibrium point or diverges to $\pm\infty$. Following a single trajectory creates a 'cobweb'. The sequence $x_1, x_2, x_3, \dots$ is called an <u>orbit</u> starting from x_1.

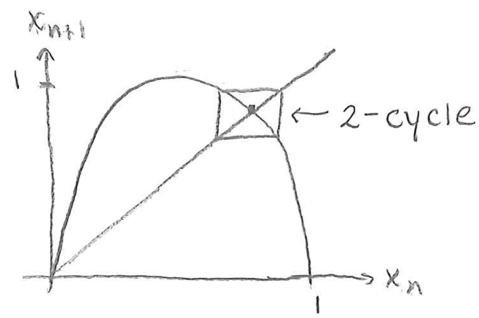
<u>Example</u>: $x_{n+1} = \cos(x_n)$



$x^* = \cos(x^*)$

$\rightarrow$ x^* is transcendental

<u>Note</u>: we could construct a map that travels between two values that don't appear to be equilibria. These are called <u>n-cycles</u> where n represents the period of the cycle.



Back to example: (logistic map)

Based on the cobweb plot for a certain α, it appears that the $x^*=0$ equilibrium is unstable and the $x^*=1-\frac{1}{\alpha}$ equilibrium is stable. We can verify this analytically by looking at the eigenvalue (aka the multiplier) associated with each equilibrium:

$$x_{n+1} = \alpha x_n(1-x_n) = \alpha x_n - \alpha x_n^2$$

The linearized equation:

$$\tilde{x}_{n+1} = f'(\tilde{x}_n)\tilde{x}_n = (\alpha - 2\alpha \tilde{x}_n)\tilde{x}_n$$

At $x^*=0$:

$$f'(0)=\alpha \quad \rightarrow \quad \text{stable if } |\alpha|<1$$

At $x^*=1-\frac{1}{\alpha}$:

$$f'\left(1-\tfrac{1}{\alpha}\right) = \alpha - 2(\alpha-1) = 2-\alpha$$

$$\rightarrow \text{stable if} \quad |2-\alpha|<1 \rightarrow -1<2-\alpha<1$$

$$\rightarrow -3<-\alpha<-1$$

$$\rightarrow 3>\alpha>1$$

(Mathematica demo for varying α)

what happens for $\alpha>3$? E.g. $\alpha=3.25$

We get a 2-cycle:

$$x_{n+2} = x_n$$

$$= \alpha x_{n+1}(1-x_{n+1}) = \alpha\left[\alpha x_n(1-x_n)(1-\alpha x_n(1-x_n))\right]$$

$$= \alpha^2 x_n(1-x_n) - \alpha^3 x_n^2(1-x_n)^2$$

Find x^*:

$$x_n\left[1-\alpha^2 x_n(1-x_n)-\alpha^3 x_n^2(1-x_n)^2\right]=0$$

$$x_n(1-\alpha+\alpha x_n)\left[1+\alpha-\alpha(1+\alpha)x_n+\alpha^2 x_n^2\right]=0$$

$$\rightarrow x_1^*=0\ , \quad x_2^*=\frac{\alpha-1}{\alpha}=1-\frac{1}{\alpha}$$

$$x_{3,4}^*=\frac{\cancel{\alpha}(1+\alpha)\pm\sqrt{\alpha^2(1+\alpha)^2-4\alpha^2(1+\alpha)}}{2\alpha^{\cancel{2}}}$$

$$=\frac{1+\alpha\pm\alpha\sqrt{1+2\alpha+\alpha^2-4-4\alpha}}{2\alpha} \qquad \alpha^2-2\alpha-3=(\alpha+1)(\alpha-3)$$

$$=\frac{1+\alpha\pm\alpha\sqrt{(\alpha+1)(\alpha-3)}}{2\alpha}$$

<u>Note</u>: $x_{3,4}^*$ is real for $\alpha>3$. For $x_{3,4}^*$ to be stable, we need $3<\alpha<1+\sqrt{6}$.

we can see this by linearizing the period-2 map,

$$x_{n+2}=f^2(x_n)=\alpha^2 x_n(1-x_n)\left[1-\alpha x_n(1-x_n)\right]$$

$$\rightarrow \lambda=\left.\frac{df^2}{dx_n}\right|_{x_{3,4}^*}=\alpha^2\Big[(1-x_n)(1-\alpha x_n(1-x_n))-x_n(1-\alpha x_n(1-x_n))$$

$$\left.+\,x_n(1-x_n)(-\alpha(1-2x_n))\Big]\right|_{x_n=x_{3,4}^*}=p,q$$

This is messy. Let's try another route:

$$x_{n+1}=\frac{d}{dx}f(f(x_n))x_n=f'(\underbrace{f(p)}_{q})f'(p)=f'(q)f'(p)$$

Since $f'(p)=\alpha(1-2p)$ and $f'(q)=\alpha(1-2q)$, we have

21/3

$$x_{n+1} = [\alpha(1-2p)][\alpha(1-2q)]\, x_n$$

$$= \alpha^2(1-2p-2q+4pq^2)\, x_n$$

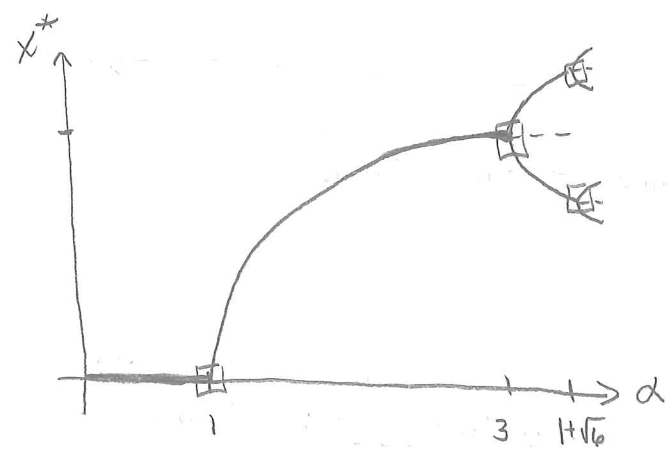
where p and q are both equili. points of the period-2 map, $f^2(x_n)$. Plugging in $p = x_3^*$ and $q = x_4^*$, we get

$$x_{n+1} = \alpha^2\left[1 - \frac{2(\alpha+1)}{\alpha} + \frac{4(\alpha+1)}{\alpha^2}\right] x_n$$

$$= 4 + 2\alpha - \alpha^2$$

So the 2-cycle is stable if $|4+2\alpha-\alpha^2| < 1 \rightarrow 3 < \alpha < 1+\sqrt{6} \approx 3.45$

We can visualize these changes in stability using a plot of the equilibria as the parameter α varies. This is called a bifurcation diagram. Points where the behavior of equilibria changes are called <u>bifurcation points</u>



$\square$ = bifurcation points

2-cycles

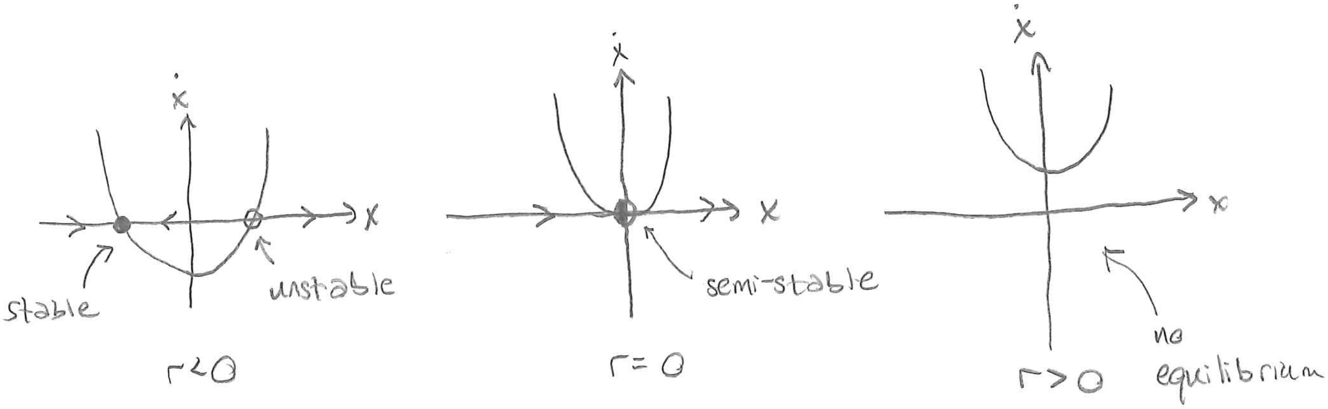
Try: 3.5 (3-cycle)

For $\alpha > 1+\sqrt{6}$, we get intermittent periodicity and chaos.

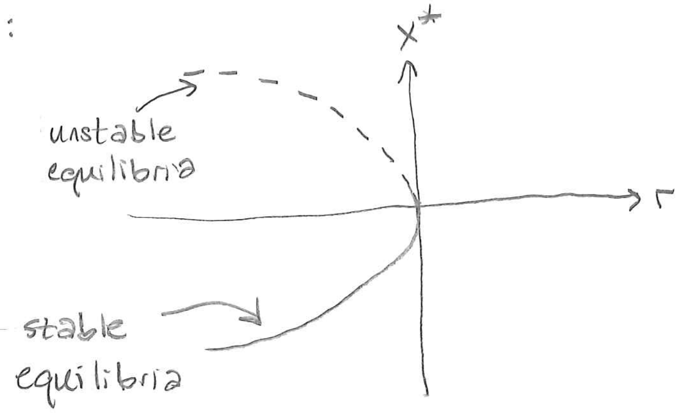
Bifurcation diagrams are not limited to discrete systems:

$$\dot{x} = r + x^2$$

Equilibria satisfy $0 = r + x^2$. If we vary r, we see the vector field (on the horizontal axis) changing with the sign of r:



We can summarize these results by only looking at a plot of $x = x^*$ vs. r:



A bifurcation point occurs at $r=0$, because at that point the two curves (also called branches) of equilibria collide and eliminate each other. This particular bifurcation is called a fold bifurcation, or a blue-sky bifurcation (because the equilibria seem to appear out of the clear blue sky!). It's also known as a saddle-node bifurcation, because it occurs between a saddle (unstable) and a stable node.

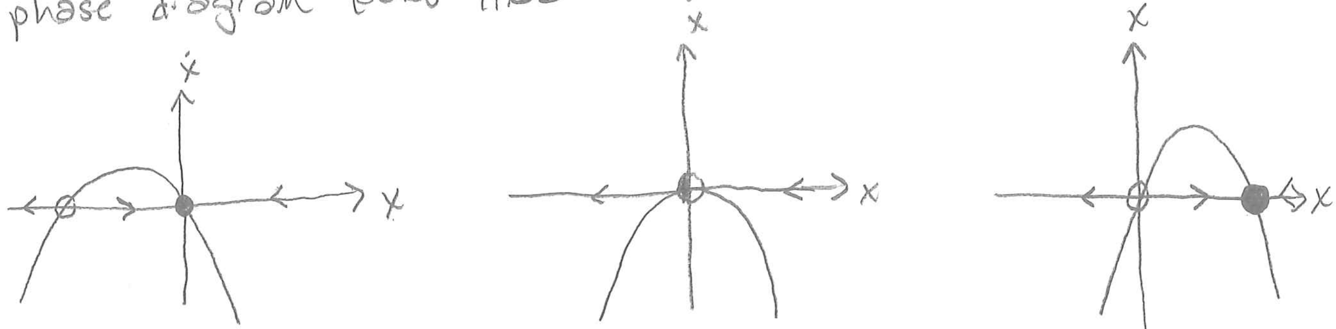
22/2

<u>Example</u>: Draw a bifurcation diagram for $\dot{x} = rx - x^2$

We can write this in steady state as

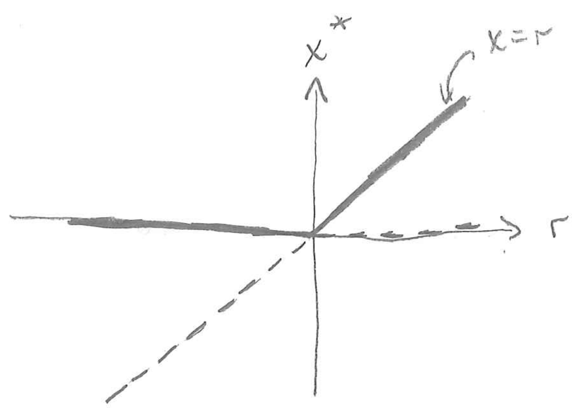
$$0 = x(r - x)$$

so we have equilibria at $x = 0$ and $x = r$. For varying r-values, the phase diagram looks like



$r < 0$ $\qquad\qquad$ $r = 0$ $\qquad\qquad$ $r > 0$

Then we can construct the bifurcation diagram by putting these phase portraits together



This is called a <u>transcritical</u> bifurcation. An example like this is the logistic equation.

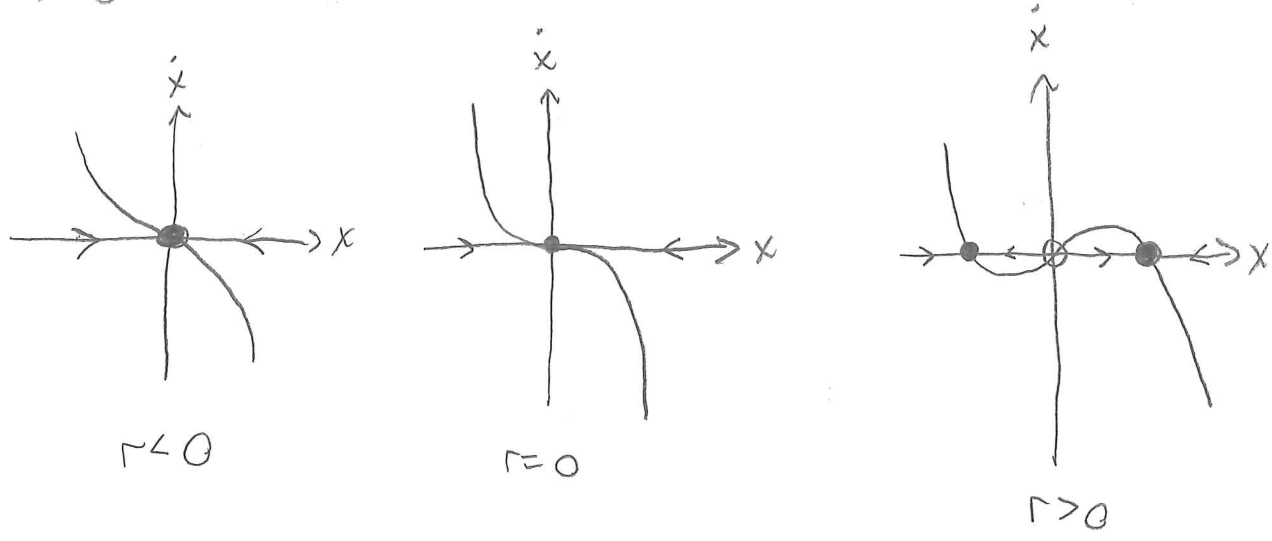
<u>Example</u>: Draw a bifurcation diagram for $\dot{x} = rx - x^3$

$$0 = x(r - x^2) = x(\sqrt{r} + x)(\sqrt{r} - x)$$

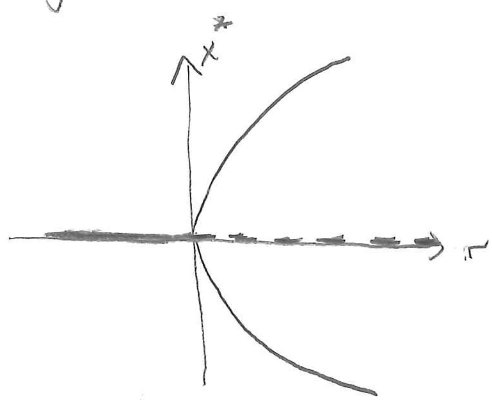
$$\rightarrow x^* = 0, \pm\sqrt{r}$$

22/3

Phase diagram:



Bifurcation diagram:



This is called a <u>pitchfork</u> bifurcation (because it looks like a pitchfork, Ψ).

A natural question may be, what does a bifurcation look like in higher dimensions?

<u>Example</u>: Romeo and Juliet (or, more generally, Lovers R and J)
This is an example in Section 5.3 of Strogatz.

Consider two lovers, R and J. J's love is fickle: the more R loves J, the more J wants to run away. But when R gets discouraged and backs off, J's love rekindles.

R tends to echo J: R loves J when J reciprocates, and R grows disinterested when J runs away. What is the fate of this relationship?

$$\dot{R} = aJ$$
$$\dot{J} = -bR$$

Since the Jacobian is $\begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix} \rightarrow \tau = 0,\ \Delta > 0$, the only equilibrium point is a center. So their relationship is fated to run an endless cycle.

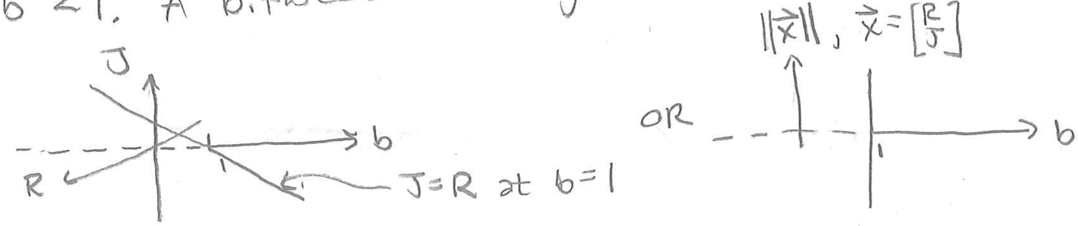
What if we described their relationship as

$$\dot{R} = -R + bJ \qquad , \qquad b > 0$$
$$\dot{J} = bR - J$$

What could be the situation described here? One possibility is that each is cautious to a certain extent, but each is attracted to the other's advances.

Now the Jacobian is $\begin{bmatrix} -1 & b \\ b & -1 \end{bmatrix}$, so $\tau = -2$ and $\Delta = 1 - b^2$

So this is a saddle if $1 - b^2 < 0 \rightarrow b^2 > 1$ and a stable node if $b^2 < 1$. A bifurcation diagram would look like



Chapter 6: Simulation of Dynamical Systems

Simulating discrete time dynamical systems simply requires us to iterate the system to create our solution:

$$\vec{x}_{n+1} = \vec{f}(\vec{x}_n) \quad , \quad \vec{x}(0) = \vec{x}_0$$

↳ Iteration: start with $\vec{x}_0$

for $n = 0$ to N ← some ending time

$\vec{x}_{n+1} = f(\vec{x}_n)$

end

For continuous time dynamical systems, we need to 'discretize' the derivative:

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}(t), t), \quad \vec{x}(0) = \vec{x}_0$$

(when we analyzed dynamical systems using 'eigenvalue methods' we didn't allow for explicit dependence of $\vec{f}$ on time (it would change our analysis), but when we solve $\frac{d\vec{x}}{dt}$ numerically we can let $\vec{f}$ depend on t explicitly)

The main idea is:

① We want to approximate $\vec{x}(t)$ on a time interval $[0, T]$.

② Introduce a time step Δt and discretize this interval.

$$0 = t_0 < t_1 < t_2 < \cdots < t_N = T$$

So $t_{i+1} = t_i + \Delta t$, and $N = \frac{T}{\Delta t}$

③ For clarity, let's call $\vec{y}(t_i)$ the discrete approximation to $\vec{x}(t_i)$

There are several possible ways we could discretize our derivative.

The Euler Method:

This is the discretization you may have seen most frequently before:

$$\frac{d\vec{x}}{dt}(t_i) \approx \frac{\vec{x}(t_i+\Delta t) - \vec{x}(t_i)}{\underbrace{\Delta t}_{\text{the book: } h}}$$

so writing the discrete approximation as $\vec{y}(t_i)$, we have

$$\frac{\vec{y}(t_i+\Delta t) - \vec{y}(t_i)}{\Delta t} = \vec{f}(\vec{y}(t_i), t_i) \to \vec{y}(t_i+\Delta t) = \vec{y}(t_i) + \Delta t \vec{f}(\vec{y}(t_i))$$

$$\to \boxed{\vec{y}_{i+1} = \vec{y}_i + \Delta t \vec{f}(\vec{y}_i, t_i)}, \quad \vec{y}_0 = \vec{x}_0$$

Forward Euler Method

We call this the Forward Euler Method because we are always looking forward in time; we call this an explicit method/scheme. I.e.,

$$\vec{y}_i \to \vec{y}_{i+1} \quad / \quad \vec{y}_{i+1} = (\text{something that does not depend on } \vec{y}_{i+1})$$

We can also look backward in time:

$$\frac{d\vec{x}}{dt}(t_{i+1}) \approx \frac{\vec{x}(t_{i+1}) - \vec{x}(t_{i+1} - \Delta t)}{\Delta t}$$

$$\to \frac{\vec{y}(t_{i+1}) - \vec{y}(t_i)}{\Delta t} = \vec{f}(\vec{y}(t_{i+1}))$$

$$\to \boxed{\vec{y}_{i+1} = \vec{y}_i + \Delta t \vec{f}(\vec{y}_{i+1}, t_{i+1})}, \quad \vec{y}_0 = \vec{x}_0$$

Backward Euler Method

Now the scheme is <u>implicit</u>, because $\vec{y}_{i+1}$ is given as an implicit function of $\vec{y}_i$.

To use Backward Euler Method, this requires a 'Newton solve', i.e. apply Newton's Method to solve the implicit relation,

$$0 = \vec{y}_i + \vec{f}(\vec{y}_{i+1}) - \vec{y}_{i+1}$$

Why do this extra work? Because the Backward Euler Method is 'more stable'. We'll talk about the stability of methods next time.

While we're discussing improvements to the Forward Euler Method, we should note that we can 'do much better' than both of these methods. First, we need to introduce the idea of the <u>order</u> of a method.

The exact solution $\vec{x}(t_i)$ can be written in approximation form by introducing an error term, τ:

$$\vec{x}_{i+1} = \vec{x}_i + \Delta t f(\vec{x}_i, t_i) + \Delta t \tau$$

Solving for τ:

$$\Delta t \tau = \vec{x}_{i+1} - \vec{x}_i - \Delta t f(\vec{x}_i, t_i)$$

$$= \vec{x}(t_i + \Delta t) \rightarrow \text{Taylor expand}$$

terms proportional to Δt^3 or higher power

$$\approx \cancel{\vec{x}(t_i)} + \Delta t \vec{x}'(t_i) + \frac{\Delta t^2}{2}\vec{x}''(t_i) + \mathcal{O}(\Delta t^3) - \cancel{\vec{x}(t_i)} - \Delta t f(\vec{x}(t_i), t_i)$$

$$= \Delta t \underbrace{\left[\vec{x}'(t_i) - f(\vec{x}(t_i), t_i)\right]}_{=0} + \frac{\Delta t^2}{2}\vec{x}''(t_i) + \mathcal{O}(\Delta t^3)$$

$$\rightarrow \tau = \frac{\Delta t}{2}\vec{x}''(t_i) + \mathcal{O}(\Delta t^2)$$

so Forward Euler (and similarly Backward Euler) have error proportional to Δt^1. We say that these methods have order 1.

The Crank-Nicolson Method

Idea: combine Forward Euler and Backward Euler:

$$\frac{\vec{x}_{i+1} - \vec{x}_i}{\Delta t} = \frac{1}{2}\left(\vec{f}(\vec{x}_i, t_i) + \vec{f}(\vec{x}_{i+1}, t_{i+1})\right) + \mathcal{O}(\Delta t^2)$$

$$\rightarrow \vec{y}_{i+1} = \vec{y}_i + \frac{\Delta t}{2}\left(\vec{f}(\vec{y}_i, t_i) + \vec{f}(\vec{y}_{i+1}, t_{i+1})\right)$$

This is a semi-implicit, second-order method.

Example: Van der Pol oscillator

$$\ddot{x} = \lambda(1-x^2)\dot{x} - x, \quad x(0) = x_0, \quad \dot{x}(0) = v_0$$

written as a first-order system,

$$\dot{x} = v$$
$$\dot{v} = \lambda(1-x^2)v - x$$

Implement in Matlab by creating vectors for your solution and a matrix for the right-hand side: let

$$\vec{F}(\vec{x}) = \begin{pmatrix} x_2 \\ \lambda(1-x_1^2)x_2 - x_1 \end{pmatrix}$$

Note: $$J\vec{F}(\vec{x}) = \begin{pmatrix} 0 & 1 \\ -2\lambda x_1 x_2 - 1 & \lambda(1-x_1^2) \end{pmatrix}$$

Then, for Forward Euler,

$$\begin{cases} \vec{y}_0 = \vec{x}_0 \\ \vec{y}_{i+1} = \vec{y}_i + \Delta t \vec{F}(\vec{y}_i) \end{cases}$$

Backward Euler

$$\begin{cases} \vec{y}_0 = \vec{x}_0 \\ \vec{y}_{i+1} = \left(\text{solution of } \vec{y}^* = \vec{y}_i + \Delta t \vec{F}(\vec{y}^*)\right) \end{cases}$$

solve using Newton's method, with

$$G(\vec{z}) = \vec{y}_i + \Delta t \vec{F}(\vec{z}) - \vec{z}$$

$$JG(\vec{z}) = \Delta t J\vec{F}(\vec{z}) - I$$

Pseudocode:

Start with an initial condition, $\vec{y}_0$.

Use Newton's Method to find $\vec{z}$ such that $G(\vec{z}) = 0$:

$$\vec{z}_{k+1} = \vec{z}_k - \left(J\vec{G}(\vec{z})\right)^{-1} \vec{G}(\vec{z})$$

$$= \vec{z}_k - \left(\Delta t J\vec{F}(\vec{z}_k) - I\right)^{-1} \left(\vec{y}_i + \Delta t \vec{F}(\vec{z}_k) - \vec{z}_k\right)$$

Iterate until converges

from optimization notes, class 9/2

Let $\vec{y}_{i+1} = \vec{z}$

Repeat for further iterations of $\vec{y}_i$

Numerical stability

Ref: Gear, 'Numerical Initial Value Problems in Ordinary Differential Equations'

Shampine, 'Numerical Solutions of Ordinary Differential Equations'

To discuss the stability of a numerical method, it helps to keep in mind our definition of an equilibrium point being stable. We want these definitions to be similar:

Definition:
An equilibrium point x_n^* of $x_{n+1} = f(x_n)$ is stable if $|\lambda| = |f'(x_n^*)| < 1$, and unstable if $|\lambda| > 1$.

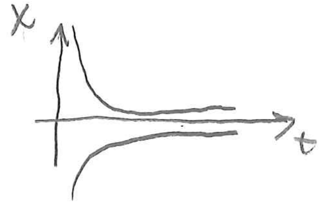
An equilibrium x^* of $\frac{dx}{dt} = f(x)$ is stable if $\lambda = f'(x^*) < 0$ (in higher dimensions, all eigenvalues λ_i of $J\vec{f}(\vec{x}^*)$ need to be negative).

Note: This is sometimes also referred to as asymptotic stability.

Definition:
A numerical method is stable if $\exists k \in \mathbb{R}$ such that if $|\Delta t| < \varepsilon$, then $|y_N - \tilde{y}_N| < k|y_0 - \tilde{y}_0|$, where $\tilde{y}_0$ is an initial perturbation and $\tilde{y}_N$ is the perturbation after N time steps.

Example:

Consider $\begin{cases} \dot{x} = 100x \\ x(0) = 10 \end{cases} \rightarrow x(t) = 10e^{-100t}$



solutions decay rapidly to zero.

Using Forward Euler, let $\Delta t = 1$. Then

$$y_1 = y_0 + \Delta t f(y_0, t_0) = 10 + 1(-100 \cdot 10) = -990$$

If we keep iterating further, we see that this solution diverges.

However, we do not conclude that Forward Euler is unstable, because if we make Δt small enough, this solution will behave as we expect and converge to zero.

Let $\Delta t = 0.0001$. Then $y_1 = 9$, and $y_i \to 0$ as $i \to \infty$.

Since the solution 'converges' if we make Δt small enough we say that FE is 'conditionally stable'. But what do we mean by convergence?

Definition: An algorithm converges if, given $\varepsilon > 0$, $\exists h$ such that if $|\Delta t_{i+1} - \Delta t_i| < h \ \forall i$, then

$$|x(T) - y_N| < \varepsilon$$

exact solution at time $t = T$

approximate solution at $t = T$

This definition is nice, but unwieldy. A more useful result is this:

<u>Theorem:</u>
If a numerical method, combined with an appropriate IVP, is stable and consistent, then it converges.

<u>Definition:</u>
A method is <u>consistent</u> if, given an IVP

$$\dot{x} = f(x,t)$$
$$x(t_0) = x_0$$

and a 1-step method $y_{n+1} = y_n + \Delta t\, \Psi(y_n, t_n, \Delta t)$, we have

$$\Psi(x, t, 0) = f(x,t)$$

In practice, these definitions can give us bounds on Δt that will ensure that our numerical solutions converge.

<u>Example:</u> $\dot{x} = \lambda x$ $\rightarrow$ $x(t) = x_0 e^{\lambda t}$ is stable if $Re(\lambda) < 0$
$x(0) = x_0$

Approximate using Forward Euler:

$$y_{i+1} = y_i + \Delta t \lambda y_i \qquad , \; y_0 = x_0$$
$$= (1 + \Delta t \lambda) y_i$$

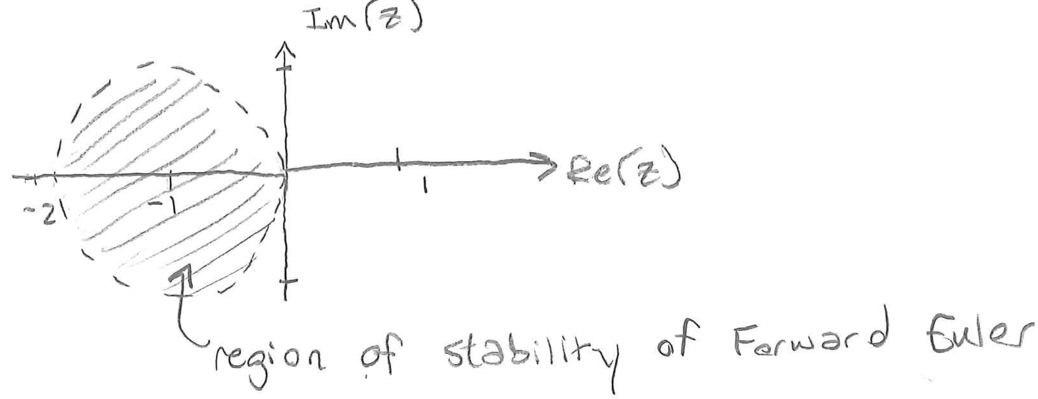
The solution is given by

$$y_i = (1 + \Delta t \lambda)^i x_0$$

This sequence remains bounded if $|1 + \Delta t \lambda| < 1$, i.e. if

$$\Delta t \lambda \in [-2, 0]$$

Remark: If we let $z = \Delta t \lambda$, we see that $|1+z|<1$ describes a disc in the complex plane:



So Forward Euler method gives a bounded sequence of approximations if the step size Δt is chosen small enough, so that

$$|1+\Delta t \lambda| < 1$$

we can interpret the Forward Euler method as a discrete-time dynamical system, which leads us to equivalent stability analysis, as we discussed on day 20:

$$\Delta \tilde{x} = \lambda \tilde{x} \qquad \leftarrow \text{linearized system}$$

s.t. $i \geq i^* \rightarrow \tilde{x}(t_i) = \tilde{x}^*$

Solution: $\Delta \tilde{x}(t_i) = \lambda \tilde{x}(t_i)$

$$\tilde{x}(t_{i+1}) - \tilde{x}(t_i) = \lambda \tilde{x}(t_i)$$

$$\tilde{x}(t_{i+1}) = (1+\lambda)\tilde{x}(t_i)$$

$$\rightarrow |\tilde{x}(t_i)| \rightarrow 0 \text{ if } |1+\lambda|<1$$

25/3

What about the stability of Crank-Nicolson?

Example: $\dot{x} = \lambda x$

$x(0) = x_0$

Approximation: $y_{i+1} = y_i + \frac{\Delta t}{2}(\lambda y_i + \lambda y_{i+1})$, $y_0 = x_0$

$$\left(1 - \frac{\Delta t \lambda}{2}\right) y_{i+1} = \left(1 + \frac{\Delta t \lambda}{2}\right) y_i$$

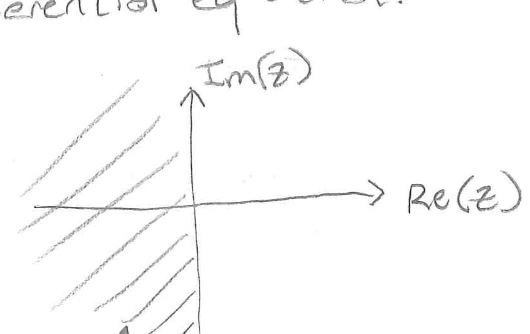
$$y_{i+1} = \frac{1 + \Delta t \lambda/2}{1 - \Delta t \lambda/2} y_i$$

Solution: $y_i = \left(\frac{1 + \Delta t \lambda/2}{1 - \Delta t \lambda/2}\right)^i x_0$

This sequence remains bounded if

$$\left| \frac{1 + \Delta t \lambda/2}{1 - \Delta t \lambda/2} \right| \leq 1 \iff \Delta t \lambda \leq 0$$

If we let $z = \Delta t \lambda$, this implies that Crank-Nicolson is stable if $Re(z) \leq 0$. This is very reminiscent of the stability requirement for an equilibrium of the continuous differential equation.



region of stability for Crank-Nicolson

Now, for backward Euler: $\dot{x} = \lambda x$

$x(0) = x_0$

we have

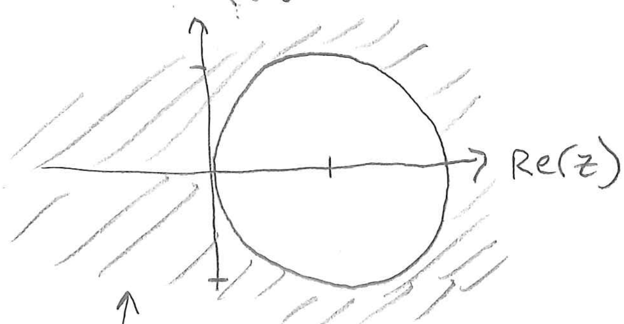
$$y_{i+1} = y_i + \Delta t \lambda y_{i+1}, \quad y_0 = x_0$$

$$(1 - \Delta t \lambda) y_{i+1} = y_i$$

$$y_{i+1} = \frac{1}{1 - \Delta t \lambda} y_i$$

$$\rightarrow \quad y_i = \frac{1}{(1 - \Delta t \lambda)^i} x_0$$

This is bounded if $\frac{1}{|1 - \Delta t \lambda|} \leq 1 \iff |1 - \underbrace{\Delta t \lambda}_{:= z}| \geq 1$



region of stability for Backward Euler

<u>Example:</u> $\dot{x} = -100x$, $x_0 = 1$ $\quad \rightarrow \quad x(t) = e^{-100t}$

Forward Euler: need $\Delta t \lambda = -100 \Delta t \in [-2, 0]$

$$\rightarrow -100 \Delta t \geq -2 \quad \rightarrow \quad \Delta t \leq \frac{1}{50}$$

Backward Euler: $1 - \Delta t \lambda \geq 1 \quad \rightarrow \quad 100 \Delta t \geq 0 \quad \rightarrow$ fulfilled $\forall \Delta t$

Crank-Nicolson: $-100 \Delta t \leq 0 \quad \rightarrow$ fulfilled $\forall \Delta t$

Example: $\dot{x} = \underbrace{100}_{\lambda} x \rightarrow x(t) = e^{100t}$

$x_0 = 1$

Forward Euler: $\{y_i\}$ bounded if $|1+\lambda\Delta t| < 1 \rightarrow$ want $|1+\lambda\Delta t| > 1$

$\lambda\Delta t > 0 \;\; \forall \Delta t \rightarrow |y_i| \rightarrow \infty$ ✓

Crank Nicolson: $\{y_i\}$ bounded if $\left|\frac{1+\lambda\Delta t/2}{1-\lambda\Delta t/2}\right| < 1 \rightarrow \lambda\Delta t < 0$

$\lambda\Delta t < 0 \;\; \forall \Delta t \rightarrow |y_i| \rightarrow \infty$ ✓

Backward Euler: $\{y_i\}$ bounded if $\left|\frac{1}{1-\lambda\Delta t}\right| < 1 \rightarrow |1-\lambda\Delta t| > 1$

want $|1-\lambda\Delta t| < 1$

$-1 < 1-\lambda\Delta t < 1 \rightarrow -2 < -\lambda\Delta t < 0 \rightarrow \lambda\Delta t \in (0,2)$

$\rightarrow \lambda\Delta t < 2$

$\rightarrow |y_i| \rightarrow \infty$

(so we need $\Delta t < \frac{1}{50}$ to get $|y_i| \rightarrow \infty$)

The Runge-Kutta Method:

(This is the method that ode45 in Matlab uses)

Pseudocode for its implementation can be found on p. 216 of Meerschaert

Runge-Kutta methods are a class of methods where the approximation of the ODE's RHS is given as a weighted average. The most widely-known Runge-Kutta method is RK4, or 4th-order Runge Kutta.

Given an IVP,

$$\dot{x} = f(x,t) \quad , \qquad x(t_0) = x_0 ,$$

the approximation is determined by

$$y_{i+1} = y_i + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

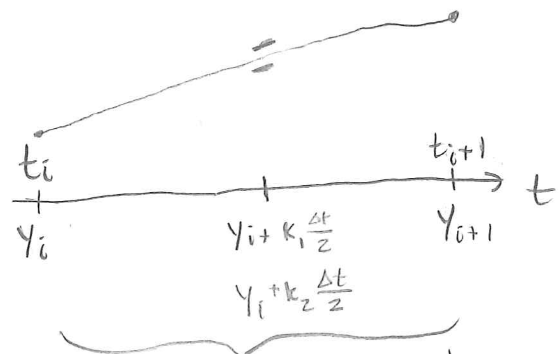
where

$$k_1 = f(y_i, t_i)$$

$$k_2 = f\left(y_i + k_1 \frac{\Delta t}{2}, t_i + \frac{\Delta t}{2}\right)$$

$$k_3 = f\left(y_i + k_2 \frac{\Delta t}{2}, t_i + \frac{\Delta t}{2}\right)$$

$$k_4 = f(y_i + k_3 \Delta t, t_i + \Delta t)$$



looks similar to Simpson's rule for numerical integration (& exactly is if f does not depend on y)

So we've split up the space between each y_i and y_{i+1} into four increments, estimated the slope at each of them using f, and used the weighted average to define y_{i+1}. Here, the two midpoints are given greater weight.

RK4 is a 4th order multi-step method. We can verify this by once again looking at the lowest power of Δt in the truncation error, τ, from

$$\frac{\Delta x}{\Delta t} = \Psi(x,t) + \tau$$

$$= \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) + \tau$$

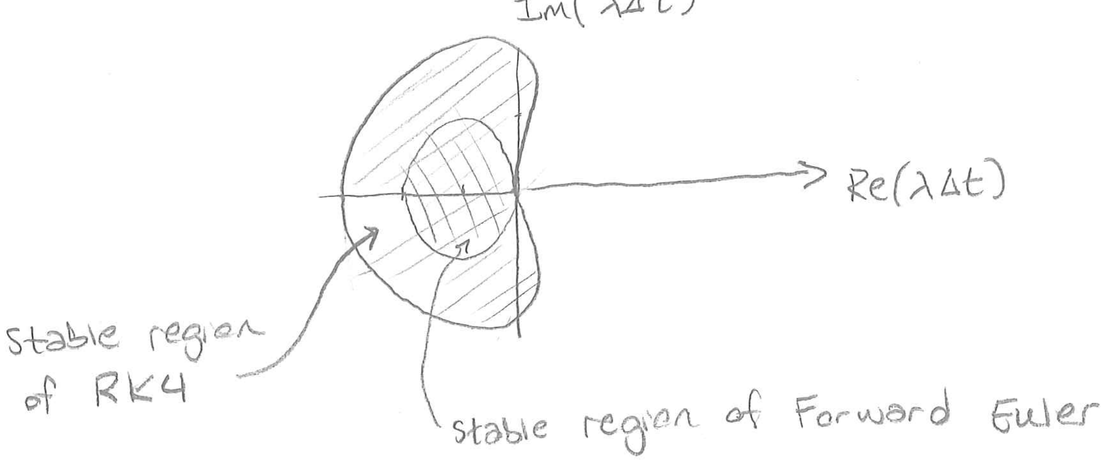
$$\rightarrow x_{i+1} = x_i - \frac{\Delta t}{6}\left(k_1(x_i,t_i) + 2k_2(x_i,t_i) + 2k_3(x_i,t_i) + k_4(x_i,t_i)\right) + \Delta t \tau$$

$\vdots$

$$\rightarrow \tau \sim \mathcal{O}(\Delta t^4)$$

what are the restrictions on Δt for RK4?

For $\dot{x} = \lambda x$, we get



Section 6.3: Chaos:

In homework, we saw an example of a discrete chaotic system, where the analogous continuous system was non-chaotic. This can happen fairly easily - even in our familiar whale system - as simple discrete systems are often chaotic.

Meerschaert goes through one example of this, where it appears that if you discretize the whale competition model with a large enough time step, you get chaotic behavior:

$$\Delta\vec{y}_i = \Delta t \vec{f}(\vec{y}_i, t_i)$$

$$\Rightarrow \quad y_{1,i+1} = y_{1,i} + \Delta t\left[0.05 y_{1,i}\left(1 - \frac{y_{1,i}}{150{,}000}\right) - \alpha y_{1,i} y_{2,i}\right]$$

$$y_{2,i+1} = y_{2,i} + \Delta t\left[0.08 y_{2,i}\left(1 - \frac{y_{2,i}}{400{,}000}\right) - \alpha y_{1,i} y_{2,i}\right]$$

for $\Delta t = 37$:

We can get chaotic behavior in some continuous-time systems even with a small time step.

<u>Example</u>: Lorenz system

$$\dot{x}_1 = -\sigma x_1 + \sigma x_2$$

$$\dot{x}_2 = -x_2 + r x_1 - x_1 x_3 \qquad , \sigma = 10, \quad b = \frac{8}{3}$$

$$\dot{x}_3 = -b x_3 + x_1 x_2$$

Equilibrium solutions:

$$0 = -\sigma x_1 + \sigma x_2 \qquad (1)$$

$$0 = -x_2 + r x_1 - x_1 x_3 \qquad (2)$$

$$0 = -b x_3 + x_1 x_2 \qquad (3)$$

$\rightarrow (x_1^*, x_2^*, x_3^*) = (0, 0, 0)$

$\rightarrow$ for $r > 1$, $(x_1^*, x_2^*, x_3^*) = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$

$(1) \rightarrow x_1 = x_2$

$(3) \rightarrow b x_3 = x_1^2 \rightarrow x_3 = \dfrac{x_1^2}{b}$

$(2) \rightarrow 0 = -x_1 + r x_1 - \dfrac{x_1^3}{b}$

$\xrightarrow[x_1 \neq 0]{} 0 = -1 + r - \dfrac{x_1^2}{b}$

$x_1^2 = b(r-1)$

$x_1 = \pm\sqrt{b(r-1)}$

Stability of the origin:

Let

$$J = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}$$

Eigenvalues are given by $\det(J - \lambda I) = 0$:

$$\rightarrow \begin{vmatrix} -\sigma-\lambda & \sigma & 0 \\ r & -1-\lambda & 0 \\ 0 & 0 & -b-\lambda \end{vmatrix} = (-b-\lambda)\Big[\underbrace{(-\sigma-\lambda)(-1-\lambda)-\sigma r}_{\lambda^2+\lambda+\sigma\lambda+\sigma}\Big] = 0$$

$$\rightarrow (-b-\lambda)\left[\lambda^2+(1+\sigma)\lambda+\sigma(1-r)\right]=0$$

$$\rightarrow \underset{<0}{\lambda_1 = -b}, \quad \lambda_{2,3} = \frac{-1-\sigma \pm \sqrt{(1+\sigma)^2-4\sigma(1-r)}}{2}$$

If $1-r<0 \rightarrow r>1$, then

$$\underbrace{(1+\sigma)^2}_{>0} \underbrace{-4\sigma(1-r)}_{>0} > (1+\sigma)^2$$

so

$$\lambda_{2,3} > \frac{-(1+\sigma)\pm(1+\sigma)}{2} = 0 \quad \text{or} \quad \frac{-2(1+\sigma)}{2}$$

so for $r>1$, the origin is unstable

For $r<1$,

$$(1+\sigma)^2-4\sigma(1-r) < (1+\sigma^2) \rightarrow \text{origin stable}$$