Below are suggestions of projects. You also have the option to choose your own project, but please have it approved by the instructor if you do. After teams have been formed, please notify me with your choice of project (hillk@umn.edu), indicating a first and second choice in case another team has already chosen your first choice. Project choices will be limited to two teams per project.

Project Garden Supply

A garden supply store discovers that some of their plants are being killed by a rare fungus. They suspect that this fungus is being transmitted from plant to plant by the customers who examine the plants. There are two groups of plants: those susceptible (S) to the fungus , and those infected with the fungus (I). The size of the susceptible group increases by the constant rate r of new plants that are brought into the store, and decreases by interactions, via customer contact, between susceptible plants and and infected plants, with an interaction coefficient α. The garden supply staff also cures β_2% of the infected plants, after which they are again susceptible to the fungus. The number of infected plants increases due to the 'interactions,' with the same interaction coefficient, and decreases due the staff curing some plants. The staff also quarantines β_1% of the infected plants, so that the size of the infected group decreases at a rate proportional to the number of infected plants. Determine the long term behavior of the plants for different values of α and β_i. Do your results change if a large number of the plants are quarantined, and eventually join the susceptible population later?

Project Salmon

Let x_n be the number of hundreds of millions of Pacific salmon at the beginning of the nth cycle. They produce a larval population y at a time t_0, which is proportional to the number of adult salmon x_n (with proportionality constant β).

What happens to the larvae? The adults cannabalize them. Then the larval population decays at a rate which is proportional to the number of interactions with the adult population, with proportionality constant α. The larvae do not remain larvae forever, and so this decay occurs only during an interval of time $t_0 < t < t_e$, which is a portion of the cycle. After this the young adults, the larvae that survived, go out to sea, and a fraction γ of them will survive. The other adults will die after breeding. So the number of salmon at the beginning of the next cycle will be the number of young adults that survive at sea.

Set up the equations which govern the salmon population, and determine if there is an equilibrium solution or cyclical behavior. Some typical parameter values are $3 < \beta\gamma < 20$, $1 < \alpha(t_e?t_0) < 10$. How could factors such as fishing, pollution, etc., affect the behavior of the population?

Project Laser

There are several models of lasers which model the interactions of the number of excited atoms N and the number of photons n in the laser field. The number of photons increases due to interactions between the photons and the excited atoms, causing emission of additional photons. This increase has interaction coefficient G. The number of photons decreases because of the escape of photons from the laser. The rate of escape is proportional to n with loss coefficient k. The number of excited atoms N decreases after emission of a photon, since they can move to a lower energy state. Thus N decreases due to interactions with the photons. The number of atoms can also decrease due to spontaneous emission, which occurs at a rate proportional to N, with rate coefficient f. The number of atoms increases at a constant rate p, which is the pump constant.

Use the description above to create a preliminary model for a laser. What is the behavior of the system for different parameter values and for different time intervals? One assumption that is sometimes used is that the number of excited atoms is at steady state, while the number of photons is not. How does this assumption affect the results? Under what conditions is this a reasonable assumption?

The Maxwell-Bloch equations provide a particular model for a laser. These equations describe the dynamics of the electric field E, the mean polarization P of the atoms, and the population inversion D:

$$\begin{aligned}
\dot{E} &= \kappa(P - E) \\
\dot{P} &= \gamma_1(ED - P) \\
\dot{D} &= \gamma_2(\lambda + 1 - D - \lambda EP),
\end{aligned}$$

where κ is the decay rate in the laser cavity due to beam transitions, γ_1 and γ_2 are decay rates of the atomic polarization and population inversion, respectively, and λ is the pumping energy parameter. The parameter λ may be positive, negative, or zero; all the other parameters are positive.

Create a simulation for Maxwell-Bloch model, and explore the behavior of the model numerically, using a variety of different parameter sets. What do you observe?

These equations are similar to the famous Lorenz equations, and can exhibit chaotic behavior for certain parameter sets. However, many practical lasers do not operate in the chaotic regime. In the very simplest case, γ_1, $\gamma_2 \gg \kappa$, and then P and D relax rapidly to steady values, and hence their time variation may be eliminated.

Assume that $\dot{P} \approx 0$ and $\dot{D} \approx 0$. Express P and D in terms of E, and thereby derive a first-order equation for E. Determine all steady states of the equation for E and determine their stability. Summarize the results in a bifurcation diagram.

Project Budworm

The size N of a budworm population is assumed to follow a logistic growth model, with a growth rate R and carrying capacity K, minus a death rate due to predation, typically by birds. The death rate is dependent on the size of the population N. When N is small, predation increases slightly with N, until N reaches a critical value. Above this critical value, the predation rate increases sharply with N until it saturates at the maximum predation level P. Give a reasonable model for the budworm population. Use this model to predict outbreak and refuge (small population) levels for the budworm population. Discuss the sensitivity of your results to the function you?ve chosen to model predation.

Project Energy Balance

Complex climate models, like those discussed in the Intergovernmental Panel on Climate Change (IPCC) reports, arose from early investigations of simple dynamical models of planetary energy balance. One current question related to planetary energy balance is whether there may be a 'tipping point' in Arctic temperatures with increased greenhouse gases, beyond which the sea ice may melt, and may not return instantaneously if greenhouse gases are lowered again. This phenomenon is called hysteresis. Investigate the phenomenon of hysteresis in a simple model for planetary energy balance, where at discrete latitudes x_i between the equation ($x = 0$) and the North Pole ($x = 1$),

$$\dot{E}_i = S(x_i, t)(1 - \alpha) - (A + BE_i) + c(\bar{E} - E_i) + F.$$

The first term represents incoming solar radiation, where $S(x_i, t) = S_0 - S_1 x_i \cos(\omega t) - S_2 x_i^2$ is the amount of sunlight that reaches the earth, at varying latitudes and times of year, and $(1 - \alpha)$ is the albedo, or the fraction of the incoming solar radiation that gets absorbed on the surface of the planet. The second term, $(A + BE_i)$, represents the outgoing longwave radiation, where A and B are constants derived from a linear fit to observations. The third term represents heat transfer as a relaxation of the energy at each latitude to the planetary mean, $\bar{E}$. The fourth term represents a constant forcing parameter, which can be attributed to the addition of greenhouse gases. Parameter values will be given to you if you choose this project.

Solutions to this differential equation are periodic oscillations. Numerically analyze bifurcations of this system, using the minimum of $E_i(t)$ as your 'equilibrium'. What happens to your bifurcation diagram when you vary the amplitude of the periodic forcing S_1, and the rate of heat transfer, c?

Project Great Lakes

The concentration of pollution in Lake Ontario increases over a period of 20 years. During this time, an average of p million cubic meters of pollution enter the lake per day. The volume of Lake Ontario is $1,636 \times 10^9$ cubic meters, and the total outflow (water and pollution combined) is around 500 or 600 million cubic meters per day. What is the concentration of pollutant in Lake Ontario after these 20 years, assuming that the initial concentration was 5%, and $10 < p < 80$? Assume that the inflow from clean sources is sufficient to maintain a constant total volume.

After the 20 years, the government has decided to clean up Lake Ontario. Assuming that the amount of incoming pollution is completely stopped, how long will it take for Lake Ontario to have a pollution concentration of 5%?

Now include the effects of Lake Erie in the clean-up. Even though no pollution is being dumped into Lake Ontario, there is pollution entering from the outflow from Lake Erie. In fact, 5/6 of the inflow into Lake Ontario is from Lake Erie, which has an outflow of 400 to 500 million cubic meters per day. If the pollution level of Lake Erie is the same as Lake Ontario at the end of the twenty years, and it does not change, how long will it take to clean up Lake Ontario? What if the pollution level of Lake Erie varies seasonally or decreases over time? Again, assume that the total volume of Lake Ontario is constant over time, so inflow and outflow balance.

Project Chemical Reaction

Lengyel and coworkers have proposed a model of a chemical reaction, involving MA, I^-, ClO_2, I_2, and ClO_2^-. By noting that MA, I_2, and ClO_2 are essentially constant, while $X = I^-$ and $Y = ClO_2^-$ vary significantly, they reduced the model to

$$\begin{aligned} \rightarrow X, &\quad r_1 = k_1 \\ X \rightarrow Y, &\quad r_2 = k_2[X] \\ 4X + Y \rightarrow P, &\quad r_3 = k_3 \frac{[X][Y]}{\alpha + X^2} \end{aligned}$$

where r_j are rate laws, and P is some product. The corresponding differential equations are

$$\frac{d[X]}{dt} = k_1 - k_2[X] - 4k_3\frac{[X][Y]}{\alpha + [X]^2}$$
$$\frac{d[Y]}{dt} = k_2[X] - k_3\frac{[X][Y]}{\alpha + [X]^2}$$

where k_1, k_2, and k_3 are proportional to $[MA]_0$, $[ClO_2]_0$, and $[I_2]_0$, the initial values of each concentration. The model can be further simplified (and nondimensionalized) to

$$\frac{dx}{d\tau} = a - x - 4\frac{xy}{1+x^2}$$
$$\frac{dy}{d\tau} = bx - \frac{bxy}{1+x^2}$$

by introducing $x = [X]/\sqrt{a}$, $y = [Y]k_3/(k_2\alpha)$, and $k_2 t = \tau$.

What are a and b in terms of the rate parameters, k_i? Under what conditions is there an equilibrium and under what conditions is the behavior cyclical? Which parameter plays the most significant role? Express your results in terms of the original rate constants. Dimensional parameter values are available upon request.

Project Arms Race

Suppose countries X and Y are engaged in an arms race; we can assume that they receive information about each other's weapons at regularly spaced intervals. The countries are following the same strategy, which requires them to have enough weapons to inflict unacceptable damage on the other country after a sneak attack. Develop a model assuming that each country has a 'feeling of safety' criteria. For example, in order to feel safe, country X needs x_0 weapons if country Y has none. For every r weapon possessed by country Y, country X needs to add one additional weapon. A similar safety protocol is held by country Y, where it will add one weapon to its arsenal for every s weapons that X possesses.

Create a model that tracks the number of warheads each country has at each time interval. Once you establish a base model, add other factors that may influence the production or depletion of warheads (e.g. an increased 'fear of attack' component, depletion of resources, a threshold number of warheads that reflects the inevitable nuclear holocaust). All functions and parameters should be well-defined (you need to be able to justify their values) and any further assumptions should be well-explained. Although you are not limited to discussing the Cold War arms race between the U.S. and the U.S.S.R., if you choose to do so, the data set below may be helpful to compare your model results to when creating your model.

Year	US warheads	Soviet warheads
1964	6,800	500
1966	5,000	550
1968	4,500	850
1970	3,900	1,800
1972	5,800	2,100
1974	8,400	2,400
1976	9,400	3,200
1978	9,800	5,200
1980	10,000	7,200
1982	11,000	10,000

Project Blood Alcohol

The average human body eliminates 12 grams of alcohol per hour. An average man aged 21-23 in good shape, weighing K kilograms has about 0.68K liters of fluid in his body. An average woman in good shape weighing K kilograms has about 0.65K liters of fluid in her body. People in poor shape have less. (1 kilogram $\approx$ 2.2046 pounds.) In most areas, the threshold for legal driving is if your body fluids contain more than one gram of alcohol per liter of body fluids (or 0.1 gm/100 mL which is the usual way of reporting it), then you are too drunk to drive legally in most jurisdictions. Find out the level for Minnesota, and use it in this project. A blood alcohol concentration of 4.0 gm/L is likely to result in coma. A blood alcohol level of 4.5 to 5.0 gm/L is likely to result in death.

Construct a model for alcohol concentration for a hypothetical person (gender and weight decided by you). You may use a discrete-time model with a short time step (1 minute is suggested), or treat time continuously (ideally, you would do both and compare your results).

Assume that Hypothetical arrives at a party and instantaneously downs a six-pack of beer. If Hypothetical doesn't enter a coma or die, how long will it be before they can legally drive home?

Construct a more realistic manner of consuming six beers. How does this affect Hypothetical's blood alcohol concentration? You may wish to use a piecewise-defined function to model periods of drinking and non-drinking. Try this study for several different types of alcohol.

Zombie Apocalypse

The classic SIR model describes the dynamics of disease epidemics, such as the flu. In the classical model, S represents the fraction of the population that is susceptible to the disease, I is the fraction that is currently infected, and R is the fraction that has recovered from the disease. The SIR model can be modified to characterize a zombie apocalypse scenario. In this case, we can split the population into three proportions: alive (A), zombified (Z), and dead (D). A general model that assumes there is no cure, zombies can die (i.e. they waste away from lack of blood), and zombies can *zombify* alive people, can be written as the following:

$$
\begin{aligned}
\dot{A} &= -\alpha f(Z)A \\
\dot{Z} &= \alpha f(Z)A - \beta Z \\
\dot{D} &= \beta Z,
\end{aligned}
$$

where $\alpha f(Z)$ is the rate of zombification which is dependent on the current number of zombies (i.e. the more zombies around, the faster the zombification rate). Here, $A + Z + D = N$, where N is the total size of the population. You can use this fact to rewrite this as a two-dimensional system. Implement this model into and build an alternative model that ensures at least some humans survive. Possible add-ons: a cure, humans can kill zombies, birth rate, humans kill each other, etc.

Alternative study: If you would prefer a more realistic version of the SIR scenario, try formulating an SIR model for the spread of White Nose Syndrome (a current epidemic in bats) using an SIR model. White nose syndrome spread data can be provided upon request.