Due September 24, beginning of class

Instructions: Show your work. An explicit, logical, and neat presentation of each solution is required, with explanations written in complete sentences. Numbered problems are from the text.

1. Chapter 2, Problem 1 (a)-(d). You can visualize the error and relative error over time by plotting them as functions of t.

2. Chapter 2, Problem 4
   (*Hint*: For (a), you do not need to write a formal proof.)

3. Consider problem 1(a) from chapter 2. For $t_i = 1, 2, 3, 4, 5, 6$, do the following:

   (a) Use Richardson's error estimate to estimate the error $y(t_i) - y_i$, when $h = 0.05$. Compare this to the results you obtained in Problem 1 (in this project).

   (b) Calculate and plot the Richardson extrapolate $\tilde{y}_i$, and compute its error.

4. In this problem, we will explore the Theta model for simple spiking of a neuron. For some background information, neurons are cells that are electrically active and are highly sensitive to external stimulation. A simple measure of the state of a neuron is to measure the difference in voltage between the inside and outside of the neuron cell wall, which is created by an imbalance of calcium, sodium, potassium, and other ions across the cell membrane. The stimulation neurons receive are effectively electrical current injected through connections with other neurons called synapses, and which then triggers channels in the cell wall to open or close allowing transport of these ions across the cell wall and consequently altering the voltage. When the voltage spikes due to a combination of injected currents into a neuron, it is said to fire. A firing neuron causes its synapses to activate, creating injection of current into its neighboring cells, and the process repeats. Different neurons have different firing patterns, and in this project, you will investigate cells which exhibit bursting, periods of rapid firing separated by periods of quiescence.

   The Theta model for neurons can be derived from the famous Hodgkin-Huxley model. It takes the form of
   $$\frac{d\theta}{dt} = 1 - \cos\theta + (1 + \cos(\theta))I(t) \equiv f(t, \theta),$$
   where $d\theta/dt$ describes the deviation from equilibrium of the voltage potential across the neuron's cell wall and I(t) represents the injection of current due to stimulation, for example through a synapse from a connected neuron. The neuron is said to fire every time the value of θ crosses odd multiples of π.

   (a) Solve the DE analytically for the cases when $I(t) = 0$, $I(t) = 1$, and $I(t) = -1$, all with initial condition $\theta(0) = \pi/4$. Determine the behavior of each of these solutions as $t \to \infty$. (*Hint*: The trig identity $1 - \cos x = 2\sin^2(x/2)$ may be helpful.)

   (b) Solve the DE approximately using Euler's Method, with $\theta(0) = \pi/4$, $I(t) = \sin(t/5)$, and $h = 0.1$. Make sure to run your program for at least five periods of bursting activity ($t_N \approx 150$). Plot $f(t, \theta)$, which gives the voltage potential across the cell membrane $V(t) \equiv d\theta/dt$, and $I(t)$, the injected current, on the same axis. Discuss the relationship between the bursting activity and the value of injected current. How does this compare with your observations in part (a)?

   (c) Repeat part (b), but with $h = 5$. Do you get the same solution as in (b)? Why or why not?

(d) Repeat part (b), using `ode45`. Do you get the same solution? What if you use smaller tolerances for the error of the `ode45` iteration? You can do this by adding this line of code before the line where you give the `ode45` command:

```
options = odeset('RelTol',1e-4,'AbsTol',1e-6);
```

These change the adaptive time step size in `ode45` based on the relative error and the absolute error. The default values for `RelTol` and `AbsTol` are 1e-3 and 1e-6, and we should keep `RelTol` > `AbsTol`. Then at the end of your `ode45` command, you should add options in a manner like the following:

```
[t,theta] = ode45(thetaRHS,tspan,theta0,options);
```

Try varying these tolerances. What happens?

(e) Compare the error and relative error of Euler's Method with $h = 0.1$, using the approximation of $V(t)$ from `ode45` as the true solution. For selected values of t, what is the ratio by which the error decreases when h is halved?

(*Hint*: It may be helpful to use 'tspan = 0:h:150' to define tspan, instead of just giving tspan an initial and a final time as we did previously.)

(f) The neuron is said to spike whenever $\theta(t) = (2n + 1)\pi$ for integer values of n. Use your best Euler Method solution to locate all the times at which the neuron fires, using the `find` function. Plot the voltage $V(t)$ for $0 \le t \le 150$, and on the same graph plot a vertical line at each time when the neuron fires using the `line` command. How do the firing times compare with the features of the voltage plot?

(*Hint*: For the `find` function, you may get better results by finding times where $|\theta(t) - (2n + 1)\pi| <$ `tol`, where `tol` is some number less than 1.)