Due October 8, beginning of class

Instructions: Show your work. An explicit, logical, and neat presentation of each solution is required, with explanations written in complete sentences. Numbered problems are from the text, though I have made minor additions to some.

1. **Investigate the stability and error of the Backward Euler Method.**

   (a) Show that Backward Euler Method with one-point iteration, defined by the iteration formula
   $$y_{i+1} = y_i + hf(t_{i+1},\, y_i + hf(t_{i+1}, y_i)),$$
   is not absolutely stable.

   (b) As a special case in which the error of the Backward Euler Method can be analyzed directly, consider the model problem
   $$y' = \lambda y\,, \quad y(0) = 1\,, \quad t \geq 0,$$
   with λ an arbitrary real constant. The backward Euler solution of the problem is given by
   $$y_i = (1 - h\lambda)^{-i}.$$
   Following the procedure for solving Problem 4(c) in Chapter 2, show that
   $$y(t_i) - y_i = -\frac{\lambda^2 t_i e^{\lambda t_i}}{2} h + \mathcal{O}(h^2).$$

2. **Use the Backward Euler Method to solve Problem 3 of Chapter 2.**

   Consider the linear problem
   $$y'(t) = \lambda y(t) + (1 - \lambda)\cos t - (1 + \lambda)\sin t\,, \quad y(0) = 1$$
   The true solution is $y(t) = \sin t + \cos t$. Solve this problem using the *Backward Euler Method* with several values of λ and h, for $0 \leq t \leq 10$. Comment on the results.

   (a) $\lambda = -1$; $h = 0.5$, 0.25, 0.125

   (b) $\lambda = 1$; $h = 0.5$, 0.25, 0.125

   (c) $\lambda = -5$; $h = 0.5$, 0.25, 0.125, 0.0625

   (d) $\lambda = 5$; $h = 0.125$, 0.0625

3. **Chapter 3, Problem 7.**

   Consider the pendulum equation
   $$ml\frac{d^2\theta}{dt^2} = -mg\sin(\theta(t))$$
   with $l = 1$ and $g = 32.2$ ft/s^2.

   (a) When θ is near zero, we can make the Taylor series approximation $\sin\theta \approx \theta$, to first order. Verify that the general solution to the differential equation with this approximation is $\theta(t) = c_1\cos(\omega t) + c_2\sin(\omega t)$, where $\omega^2 = g/l$.

(b) For the initial conditions, choose a value in $0 < \theta(0) \le \pi/2$, and set $\theta'(0) = 0$. Use Euler's method to solve this equation as a system, given in (3.14), and experiment with various values of h to obtain a suitably small error in the computed solution. Justify your reasoning for your choice of h and provide a plot of the error for your chosen value of h.

(c) Graph t vs. $\theta(t)$, t vs. $\theta'(t)$, and $\theta(t)$ vs. $\theta'(t)$ from part (b). Does the motion appear to be periodic in time? It might be illustrative to see what happens for a few different initial conditions $\theta(0)$ and $\theta'(0) \in [0, 10]$.

(d) Redo parts (b) and (c) using the same initial condition for $\theta(0)$ and with $\theta'(0) = 11$. What do you observe?

(e) Now consider the *damped* pendulum equation,

$$l\frac{d^2\theta}{dt^2} + cl\frac{d\theta}{dt} = -g\sin\theta,$$

where $c > 0$ adds in damped motion representing resistance proportional to (angular) velocity. Set $c = 1$. Alter your Euler Method code accordingly and approximate the solution using the same initial conditions as in parts (b) and (d). Describe any changes in motion you observe between the original and damped pendulum equations.

4. **Using the indicated methods below, solve the system in Chapter 3, Problem 1.**

Consider the IVP $\vec{y}'(t) = A\vec{y}(t) + \vec{g}(t)$ with $\vec{y}(t_0) = \vec{y}_0$, where

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}, \qquad \vec{g}(t) = \begin{bmatrix} -2e^{-t} + 2 \\ -2e^{-t} + 1 \end{bmatrix}, \qquad \vec{y}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The true solution is $\vec{y}(t) = [e^{-t},\, 1]^\intercal$. Use stepsizes of $h = 0.1,\ 0.05,\ 0.025$, and solve for $0 \le t \le 10$. Use Richardson's error formula to estimate the error for $h = 0.025$.

(a) the Forward Euler Method

(b) the Backward Euler Method

5. **Chapter 4, Problem 10**

Determine whether the midpoint method

$$y_{i+1} = y_i + hf\left(t_{i+1/2},\, \frac{1}{2}(y_i + y_{i+1})\right),$$

where $t_{i+1/2} = (t_i + t_{i+1})/2$, is absolutely stable.