Due October 22, beginning of class

Instructions: Show your work. An explicit, logical, and neat presentation of each solution is required, with explanations written in complete sentences. Numbered problems are from the text.

1. **Chapter 4, Problem 7**
   Solve the equation
   $$y'(t) = \lambda y(t) + \frac{1}{1+t^2} - \lambda \tan^{-1}(t)\,, \quad y(0) = 0.$$
   The true solution is $y(t) = \tan^{-1}(t)$. Use Euler's method, the backward Euler method, and the trapezoidal method. Let $\lambda = -1$, -10, -50, and $h = 0.5$, 0.1, 0.001. Discuss the results, including both the approximations and their error. In implementing the backward Euler method and the trapezoidal method, note that the implicit equation for y_{n+1} can be solved explicitly without iteration.

2. **Chapter 4, Problem 15**
   Consider the IVP
   $$y'(t) = \lambda y\,, \quad y(0) = 1\,, \quad \lambda < 0$$
   (a) Use the Backward Euler Method formula for this IVP, $y_i = (1 - h\lambda)^{-i}$, and the corresponding formula for the Trapezoidal Method,
   $$y_i = \left(\frac{1 + \frac{1}{2}\lambda h}{1 - \frac{1}{2}\lambda h}\right)^i, \quad i \geq 0,$$
   to show that the Backward Euler Method performs better than the Trapezoidal Method with λ negative and very large.

   (b) Considering part (a) in combination with the results we derived in class, when would you use the Trapezoidal Method vs. the Backward Euler Method? Compare and contrast the strengths of each method.

3. **Chapter 5, Problem 1**

4. (a) Derive the second-order Runge Kutta methods for
   $$y_{i+1} = y_i + hF(t_i, y_i; h), \quad i \geq 0,$$
   corresponding to $b_2 = 3/4$ and $b_2 = 1$.

   (b) For the method with $b_2 = 1$, draw an illustrative graph analogous to that of Figure 5.1 for $b_2 = 1/2$. Explain the meaning of the slope of each line you draw.

   (c) Solve the problem (5.1),
   $$y' = -y + 2\cos t\,, \quad y(0) = 1, \quad t \in [0, b],$$
   whose true solution is $y(t) = \sin t + \cos t$, with the formula for $b_2 = 1$ from part (a). Choose an appropriate value of b and use $h = 0.1$. Compare your results graphically to those obtained with Heun's Method (formula (5.20), where $b_2 = 1/2$), by iterating it as well.

5. **Chapter 5, Problem 13 with RK4**

Consider the predator-prey model

$$\begin{aligned} x'(t) &= Ax(1 - By), \quad x(0) = x_0, \\ y'(t) &= Cy(Dx - 1), \quad y(0) = y_0, \end{aligned}$$

where $A = 4$, $B = 0.5$, $C = 3$, and $D = 1/3$.

(a) Show that there is a solution $x(t) = c_1$, $y(t) = c_2$, with c_1 and c_2 nonzero constants. What would be the physical interpretation of such a solution $[x(t),\, y(t)]^\intercal$?
(Hint: What are $x'(t)$ and $y'(t)$ in this case?)

(b) Solve the model with $x(0) = 3$, $y(0) = 5$, for $t \in [0, 4]$, using Runge-Kutta 4 with step size $h = 0.01$. Plot the solution in both variables, t vs. $x(t)$ and $y(t)$, as well as x vs. y.
(*Hint:* It might be helpful to reference (3.26), which is Heun's Method written for a system of DEs.)

(c) Repeat (b) for the initial values $x(0) = 3$ and $y(0) = 1$, 1.5, 1.9 in succession. Comment on the relation of these solutions to one another and to the solution of part (a).