Due November 12

Instructions: Show your work. An explicit, logical, and neat presentation of each solution is required, with explanations written in complete sentences. Numbered problems are from the text.

1. **Chapter 2, Problem 3 with Trapezoidal Method**

   Consider the linear problem

$$y'(t) = \lambda y(t) + (1 - \lambda)\cos t - (1 + \lambda)\sin t\,, \quad y(0) = 1$$

   The true solution is $y(t) = \sin t + \cos t$.

   (a) Solve this problem using the *Trapezoidal Method* with several values of λ and h, for $0 \leq t \leq 10$. Comment on the results. In your discussion, make sure to compare these results with those obtained in previous projects, using the Backward Euler Method. Do the results match your expectations?

      i. $\lambda = -1$; $h = 0.5$, 0.25, 0.125
      ii. $\lambda = 1$; $h = 0.5$, 0.25, 0.125
      iii. $\lambda = -5$; $h = 0.5$, 0.25, 0.125, 0.0625
      iv. $\lambda = 5$; $h = 0.125$, 0.0625

   (b) For $\lambda = -1$ and $h = 0.5$, calculate the Richardson error estimate and verify that it matches the error with the true solution.

2. **Third-order Adams-Bashforth Method**

   (a) Derive the Adams-Bashforth method obtained when using an explicit interpolating polynomial of degree 2.

   (b) Determine the region of absolute stability for AB3 when $\lambda \in \mathbb{R}$.

   (c) A model for the collapse of a spherical cavity in a liquid, as given in a classical text on fluid mechanics by Lighthill (1986), is given below. It is formulated as a partial differential equation, which we can reduce to an ODE using the spherical symmetry of the problem,

$$2\pi\rho a^3 \left(\frac{da}{dt}\right)^2 = \frac{4}{3}\pi(a_0^2 - a^2)P.$$

   Here, $a(t)$ is the radius of the cavity at time t, $a_0 = a(0)$, ρ is the density of the fluid, and P is the constant pressure in the neighborhood of the cavity.

   After rescaling variables and parameters to render the differential equation dimensionless, it becomes

$$\frac{dr}{dt} = -\sqrt{\frac{2}{3}\left(r^{-3} - 1\right)}, \qquad r(0) = 0.95, \qquad r \geq 0, \qquad t \in [0, 1]$$

   Use AB3 to approximate the solution to this initial value problem. Use an appropriate step size based on your prior calculations. To calculate y_1 and y_2, use one of the second-order RK methods from Chapter 5.

3. **Third-order Adams-Moulton Method**

   Consider the *knee problem*,

   $$\epsilon \frac{dy}{dx} = (1-x)y - y^2, \qquad y(0) = 1, \qquad x \in [0, 2].$$

   (a) Verify that this differential equation is stiff.

   (b) Derive the Adams-Moulton method obtained when using an implicit interpolating polynomial of degree 2.

   (c) Use AM3 to approximate the solution when $\epsilon = 10^{-4}$. Use an appropriate step size resolve the behavior that $y \approx 0$ when $x \approx 2$. What does this indicate about the step size to use for implicit methods and/or stiff problems?

   (d) Based on the differentia equation, what would you expect to happen when ϵ gets smaller? After reflecting on this, use AM3 to approximate the solution when $\epsilon = 10^{-6}$. What do you think might be happening?

4. **Method of Lines and Example 8.3**

   Consider the partial differential equation problem

   $$\begin{aligned} u_t &= u_{xx} + G(t,x), \qquad 0 < x < 1, \qquad 0 < t \leq 10 \\ u(t,0) &= d_0(t), \qquad u(t,1) = d_1(t), \qquad t \geq 0 \\ u(0,x) &= f(x), \qquad 0 \leq x \leq 1 \end{aligned}$$

   (a) Determine $G(t,x)$, $d_0(t)$, $d_1(t)$, and $f(x)$ when the true solution to the equation is given by

   $$u(t,x) = e^{-0.1t}\sin(\pi x), \qquad 0 \leq x \leq 1, \qquad t \geq 0$$

   (b) Determine the Method of Lines approximation to the PDE using the D_+D_- approximation to the spatial derivative u_{xx}. For $m = 16$, verify numerically that this system of ordinary differential equations is stiff and plot the eigenvalues as a scatter plot in the complex plane.

   (c) Use the Backward Euler Method to approximate the solution to the system derived in part (b) for $h = 0.1$ and $m = 4$, 8, and 16. Make three figures for each m: (1) a plot of the Euler Method approximation in 3 dimensions using the Matlab command `surf` or `mesh`; (2) a plot of the true solution; and (3) a plot of both the true solution and the Method of Lines approximation for $t = 0$ and a couple of other interesting times (i.e., plots of x vs. u). Discuss how your results compare.

   (*Hint:* you can create a matrix of t and x values for the true solution using the command `meshgrid`.)

5. **Chapter 8, Problem 3**

   (a) Derive the backward differentiation formula of order 2.

   (b) Using the BDF method of order 2, approximate the solution to the IVP

   $$y'(t) = \lambda y(t) + (1-\lambda)\cos t - (1+\lambda)\sin t, \qquad y(0) = 1,$$

   which has the true solution $y(t) = \sin t + \cos t$. Discuss how the approximation compares to the true solution for $h = 0.5$ and $\lambda = -1$, -10, and -50. How does your conclusion change when h is deacreased, e.g. $h = 0.1$ or $h = 0.01$?

   (*Hint:* Note that the linearity of the test equation (8.8) allows the implicit BDF equation for y_{n+1} to be solved explicitly; iteration is unnecessary.)